Optimal Approximation of Biquartic Polynomials by Bicubic Splines

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Content

I. Motivation

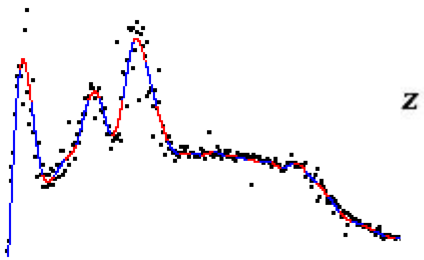
II. 2D - Quartic and Cubic Polynomials

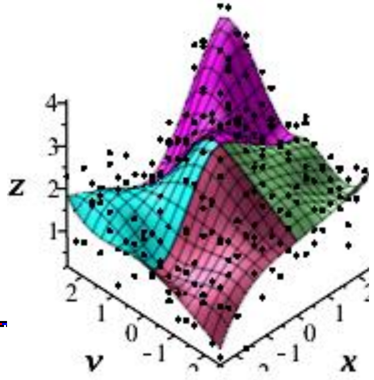
- Basic Idea
- Generalization
  - Reduced System – Computation Speedup
  - k^{th} Reduced Systems
  - Explicit Interpolating B-Splines

III. 3D - Biquartic and Bicubic P.

- Generalization
  - Reduced System – Speedup computation
- Optimality

Motivation





Areas of using smooth curves, surfaces

- Data processing, CAD, Game industry
- Virtual Caves, Robotics, Drones

Stable, Fast, Simple Techniques

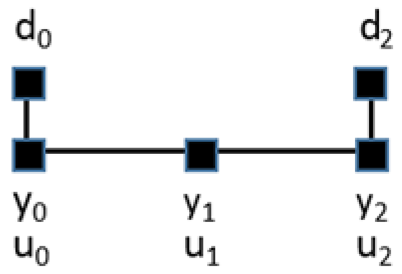
Polynomials, Splines

Aspects

- Analytical
- Geometrical
- Abstraction vs. Generalization
- Computational
- Physical

New Representation of Models, Algorithms, Implementation

Basic Idea – approximation of Quartic Polynomials



Arbitrary inner point u_1

- Uniquely defined quartic polynomial

Coincidence in 6 values ($*$)

$\Downarrow$

- Interpolating Hermite Spline is of class C^1

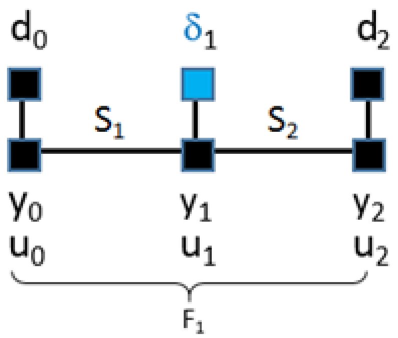
Do the coefficients of the QP influence the optimal location of u_1?

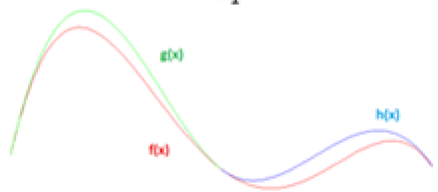
Basic Idea – Quartic Polynomial

$$\text{i)}\quad ||f||_2 = \left(\int_a^b |f(x)|^2 \, \mathrm{d}x \right)^{1/2},$$

$$\text{ii)}\quad ||f||_\infty = \max_{a \le x \le b} |f(x)|,$$

$$\text{iii)}\quad ||f||_{2,2} = \left(\int_a^b |f''(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$



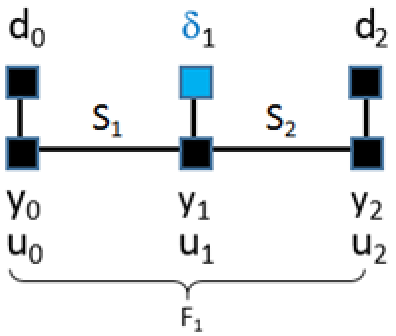


Basic Idea – Quartic Polynomial

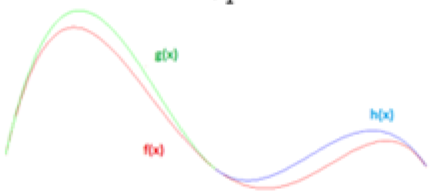
i) $$||f||_2 = \left(\int_a^b |f(x)|^2 \, \mathrm{d}x \right)^{1/2},$$

ii) $$||f||_\infty = \max_{a \le x \le b} |f(x)|,$$

iii) $$||f||_{2,2} = \left(\int_a^b |f''(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$



(*)



Theorem

The two-component bicubic C^1-class Hermite spline $S = \{g(x), h(x)\}$, defined by the eight equalities of ($*$), approximating $f(x)$ over intervals $[u_0, u_1]$ and $[u_1, u_2]$ is of class C^2, i. e.

$$g''(u_1) = h''(u_1), \tag{54}$$

if $\boxed{u_1 = \frac{u_0+u_2}{2}.}$

Török, 2013, 2015
Miňo, Buša, Török 2014

1. Uniform

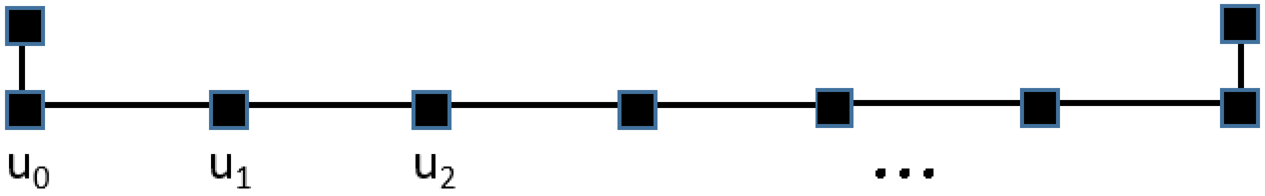
2. From Quartic

$\Rightarrow$

1. Hermite Spline $\in C^2$

2. Optimal (depends on the order of QP)

Generalization on a Uniform Grid



Problem statement (*)

Principle: the 1st derivative of the CP / QP are adjusted

$\{(u_i, y_i),\ i = 0, \dots, n+1\} \subset \mathbb{R}^2 \quad u_{i+1} - u_i = h > 0$

d_0, d_{n+1}

$$\mathbf{d} = (d_1, \dots, d_n)^\mathsf{T}$$

$$\mathbf{T}_n(1,4,1) \cdot \mathbf{d} = \boldsymbol{\varrho},$$

Stable Full tridiagonal system –
LU deccomposition

$$\begin{pmatrix} 4 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 4 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix} = \begin{pmatrix} \frac{3}{h}(y_2 - y_0) - d_0 \\ \frac{3}{h}(y_3 - y_1) \\ \frac{3}{h}(y_4 - y_2) \\ \vdots \\ \frac{3}{h}(y_n - y_{n-2}) \\ \frac{3}{h}(y_{n+1} - y_{n-1}) - d_{n+1} \end{pmatrix}$$

Reduced Tridiagonal Systems, Sequential Algorithm

Theorem 1. *The tridiagonal system*

$$\begin{bmatrix} -14 & 1 & 0 & & & \\ 1 & -14 & 1 & & & \\ 0 & 1 & -14 & & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & & & -14 & 1 \\ & & & & & \mu \end{bmatrix} \begin{bmatrix} d_2 \\ d_4 \\ d_6 \\ \vdots \\ d_{\nu-2} \\ d_\nu \end{bmatrix} = \begin{bmatrix} \frac{3}{h}(y_4 - y_0) - \frac{12}{h}(y_3 - y_1) - d_0 \\ \frac{3}{h}(y_6 - y_2) - \frac{12}{h}(y_5 - y_3) \\ \frac{3}{h}(y_8 - y_4) - \frac{12}{h}(y_7 - y_5) \\ \vdots \\ \frac{3}{h}(y_\nu - y_{\nu-4}) - \frac{12}{h}(y_{\nu-1} - y_{\nu-3}) \\ \frac{3}{h}(y_{\nu+\tau} - y_{\nu-2}) - \frac{12}{h}(y_{\nu+1} - y_{\nu-1}) + \eta d_{N+1} \end{bmatrix},$$

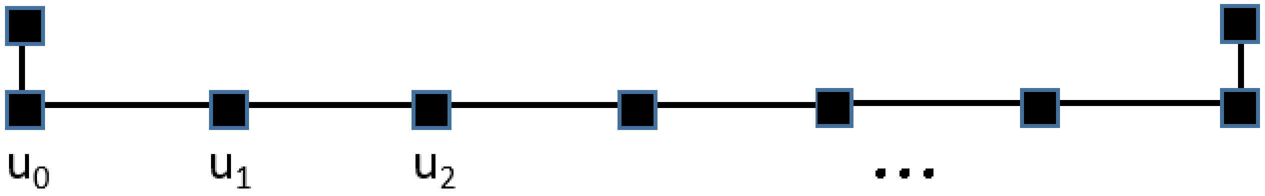
where

$$\begin{aligned} &\mu = -15,\ \tau = 0,\ \eta = 4,\ \nu = N, && \textit{if } N \textit{ is even}, \\ &\mu = -14,\ \tau = 2,\ \eta = \text{-}1,\ \nu = N-1, && \textit{if } N \textit{ is odd}, \end{aligned} \tag{8}$$

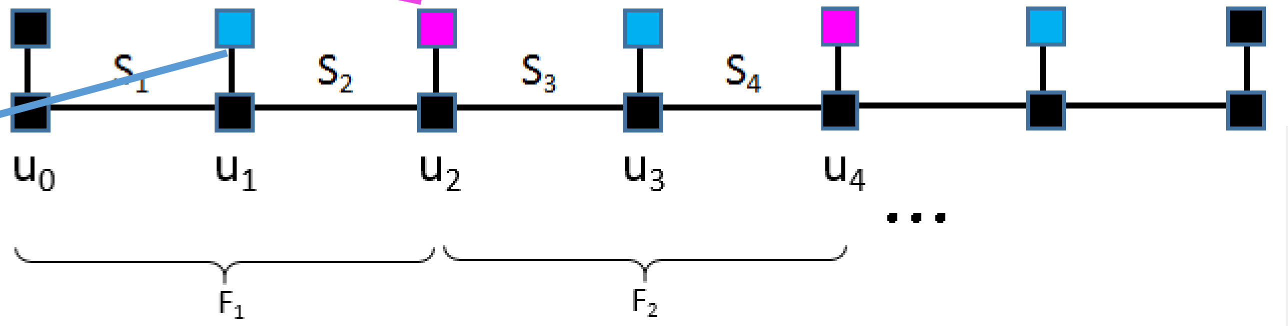
and the formula

$$d_i = \frac{1}{4}\left(\frac{3}{h}(y_{i+1} - y_{i-1}) - d_{i-1} - d_{i+1}\right),\ i = 1, 3, \ldots, \nu + \tau - 1, \tag{9}$$

grant that the second derivatives of spline components at the inner grid points are equal.



odd nodes – even indices



Full=Reduced+Rest

Speedup

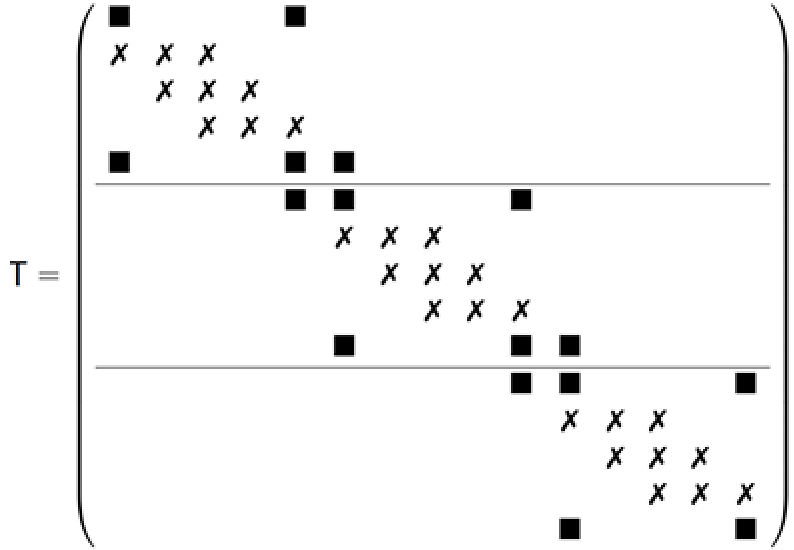
- Assessed theoretical ≈ 1.54
- Measured ≈ 1.55
  - 2 times less equations
  - Less divisions
  - Instruction Level Parallelism

$$\mathbf{T}_n(1,4,1) \cdot \mathbf{d} = \varrho,$$

$$\begin{pmatrix} 4 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 4 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 4 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 4 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix} = \begin{pmatrix} \frac{3}{h}(y_2 - y_0) - d_0 \\ \frac{3}{h}(y_3 - y_1) \\ \frac{3}{h}(y_4 - y_2) \\ \vdots \\ \frac{3}{h}(y_n - y_{n-2}) \\ \frac{3}{h}(y_{n+1} - y_{n-1}) - d_{n+1} \end{pmatrix}$$

Török, 2016, 2017

Parallel Algorithm



- Austin-Berndt-Moulton (ABM) + Reduced algorithm

CUDA

Speedup ≈ 3.5

Jadloš, Kačala, Török, 2017

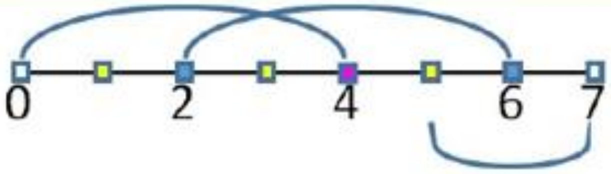
k^{th} Reduced Tridiagonal Systems

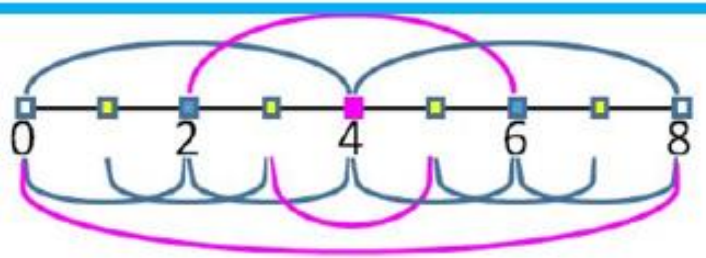
1) Geometrical - Quartic P.
2) Analytical - Spline
3) Abstract - Matrices & not only for splines

$$M^1: \quad d_0 - 14d_2 + d_4 = \frac{3}{h}(y_4 - y_0) - \frac{12}{h}(y_3 - y_1)$$

$$\begin{bmatrix} 2\,ac - b^2 & c^2 & 0 \\ a^2 & 2\,ac - b^2 & c^2 \\ 0 & a^2 & 2\,ac - b^2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix} = \begin{bmatrix} r_1 a - r_2 b + r_3 c - a^2 x_0 \\ r_3 a - r_4 b + r_5 c \\ r_5 a - r_6 b + r_7 c - c^2 x_8 \end{bmatrix}$$

$$\boxed{M^2: -d_0 + 194\,d_4 - d_8 = 3\,\frac{y0 - y8}{h} - 12\,\frac{y1 - y7}{h} + 42\,\frac{y2 - y6}{h} - 156\,\frac{y3 - y5}{h}.}$$





Explicit Interpolating Splines

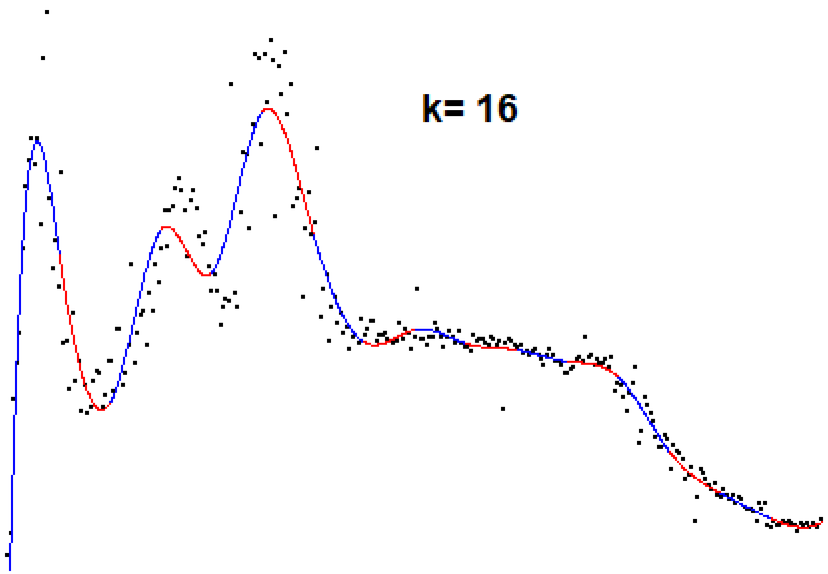
$$(d_1, \ldots, d_n)^{\mathsf{T}} = \mathbf{d} = \mathbf{T}^{-1} \cdot \varrho$$
$$= \mathbf{A} \cdot \gamma,$$
$$[s_1(x), \ldots, s_{n+1}(x)]^{\mathsf{T}} = \mathbf{S} = \mathbf{B} \cdot \gamma \text{ explicit}$$

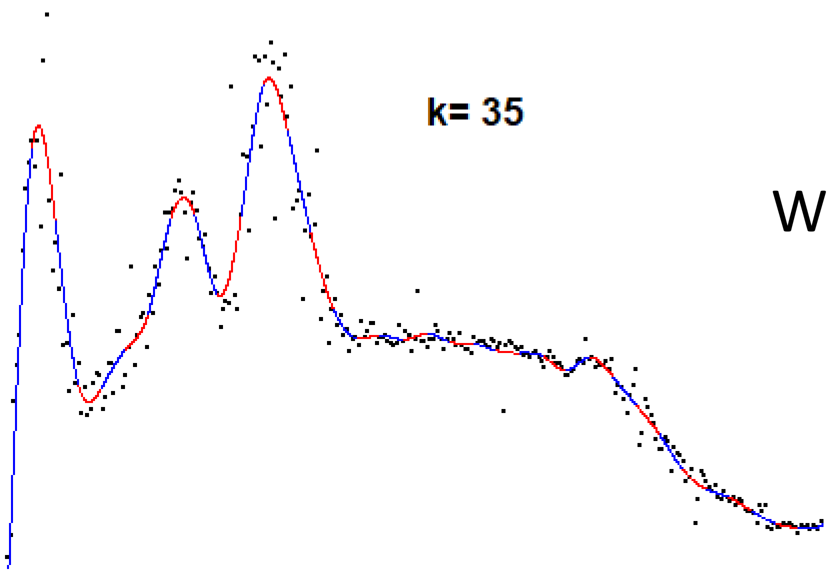
kde $\gamma = (y_0, y_1, \ldots, y_{n+1}, d_0, d_{n+1})^{\mathsf{T}}.$

Explicit Interpolating B-Splines

$$[s_1(x), \ldots, s_{n+1}(x)]^{\mathsf{T}} = \mathbf{S} = \mathbf{B} \cdot c \quad \text{explicit}$$
$$= \mathbf{S} = \mathbf{B} \cdot C \cdot \gamma \text{ explicit}$$

Popularity – Interpretability – Reparameterization



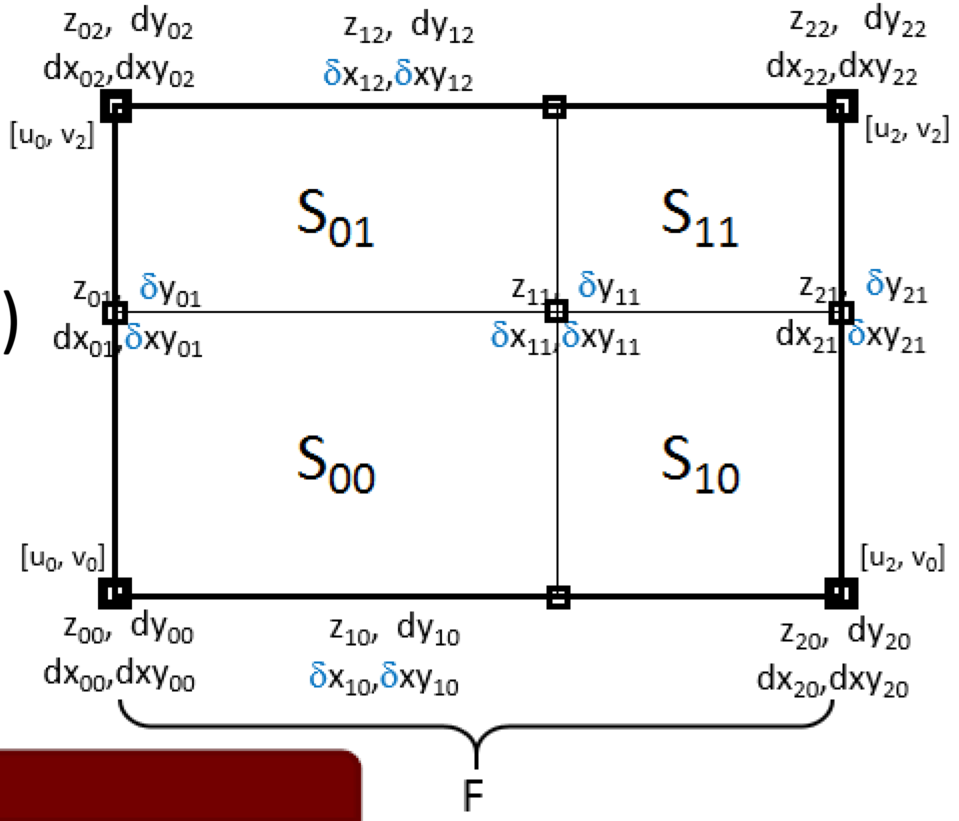


What is here interesting?

Hudák, Török, 2017

Nonuniform!

Biquartic & Bicubic Polynomials



Problem statement (*)

9
12
4
25
+ 11 = 36

Theorem

The biquartic polynomial $F(x,y)$ based bicubic Hermite spline components S_{00}, S_{10}, S_{01}, S_{11} that approximate $F(x,y)$ under conditions (*) make up a clamped bicubic spline with C^2 continuity if

$$\boxed{u_1 = \frac{u_0+u_2}{2} \quad \text{and} \quad v_1 = \frac{v_0+v_2}{2}}$$

Szabó, Török, 2014

1. Uniform
2. From Quartic

$\Rightarrow$ 1. HS $\in C^2$

No Systems of Eq.
Nothing about **optimality**

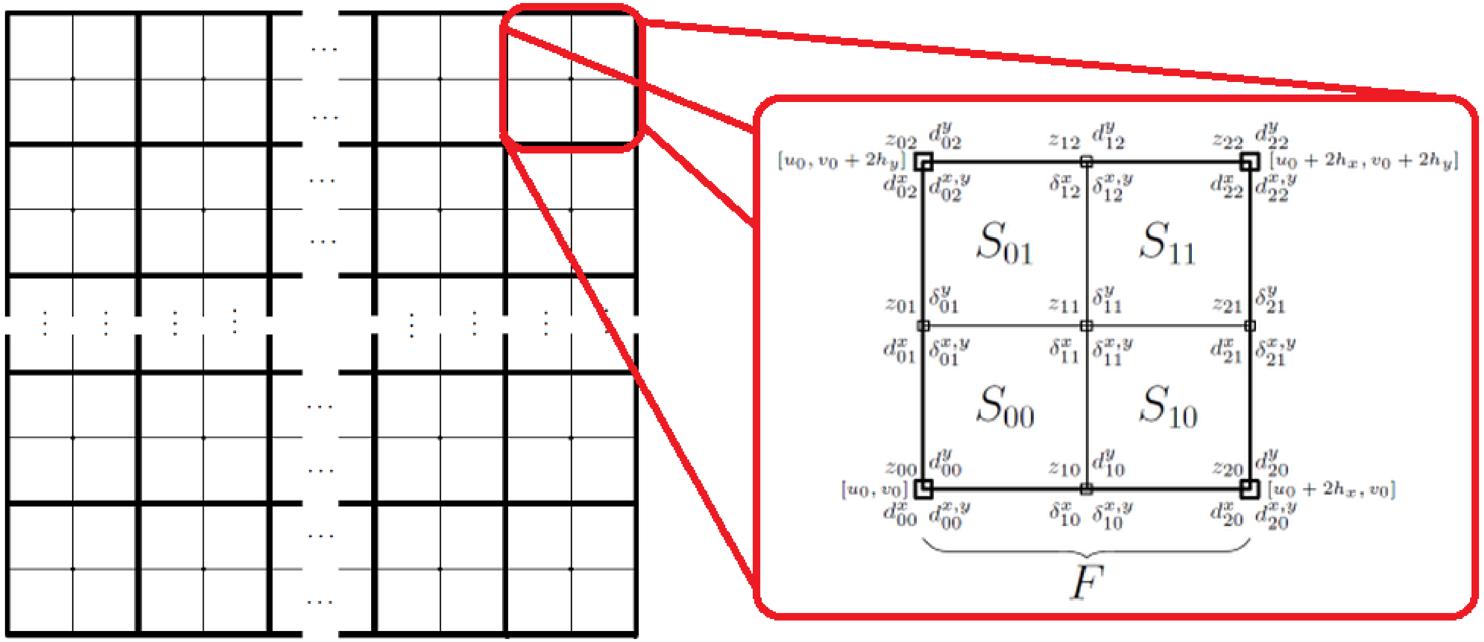
Interpol. Sp. Surfaces

- De Boor's **Full** Algorithm

- Biquartic P. based **Reduced** Algorithm

Full=Reduced+Rest

- Half size tridiagonal systems
- Explicit formulas



Uniform grid

Speedup ≈ 1.2 < 1.55 ?   Correctness vs. Time

For Cross Derivatives Intricate Explicit Formulas

Two solutions - ok

- ? return to optimality

- Miňo, Török, 2015
- V.Kačala, L.Miňo, 2017

Optimal Approximation of Biquartic Splines

Theorem

The biquartic polynomial $F(x, y)$ based bicubic Hermite spline components S_{00}, S_{10}, S_{01}, S_{11} that approximate $F(x, y)$ under conditions ($*$) make up a clamped bicubic spline with C^2 continuity if

$$u_1 = \frac{u_0 + u_2}{2} \quad \text{and} \quad v_1 = \frac{v_0 + v_2}{2}.$$

3) If $\boxed{a_{42} = a_{43} = 0 = a_{24} = a_{34},}$

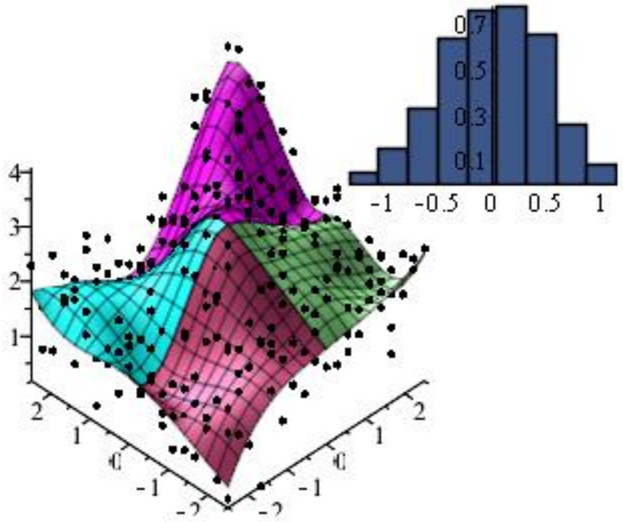
then the Hermite spline S={S_{00}, S_{10}, S_{01}, S_{11}} optimally approximates *F*(x,y) in Holladay's semi-norm.

$$\|f\|^2 = \int_{x_a}^{x_b} \int_{y_a}^{y_b} \left| \frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2} \right|^2 dy\,dx$$

Can these 4 constraints

- be embeded into a new computational algorithm that would grant optimality
- speed up the computation?

Summary



We offer

- Quartic P. based Fast Reduced Tridiagonal Systems
  - Speedup 2D: 1.55(seq), 3.5(par); 3D: 1.2(seq)
- Explicit Splines

Future Work

- Speedup of Bicubic Spline Surfaces
- Knot Detection & Sequential Smoothing
- Nonuniform Reduced Systems & Explicit Splines
- k^{th} Reduced Systems

Many Thanks

- Higher Degree Polynomials
- Knot Detection & Sequential Smoothing

116 Years of Runge Phenomenon – Uniform Grid

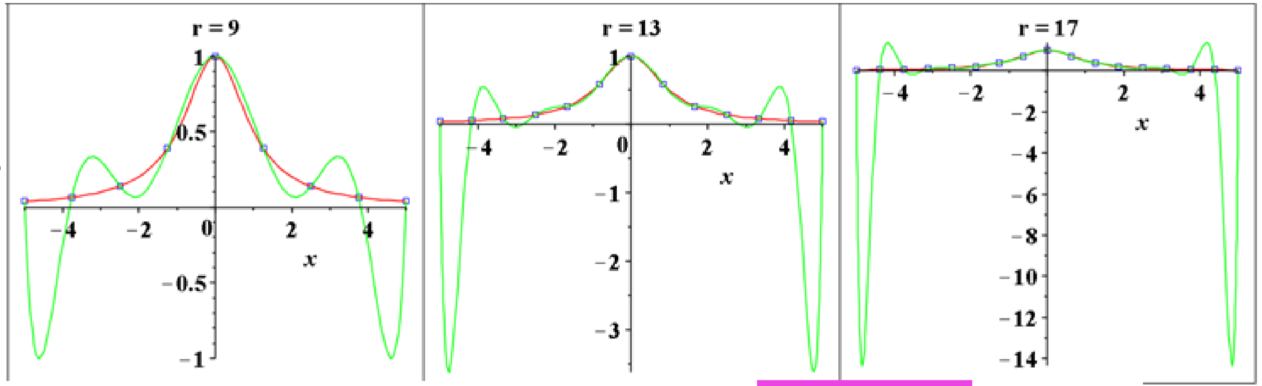
- Chebyshev
  - Oscillatoric Theorem
  - Chebyshev's Arguments
  - Chebyshevov polynóm

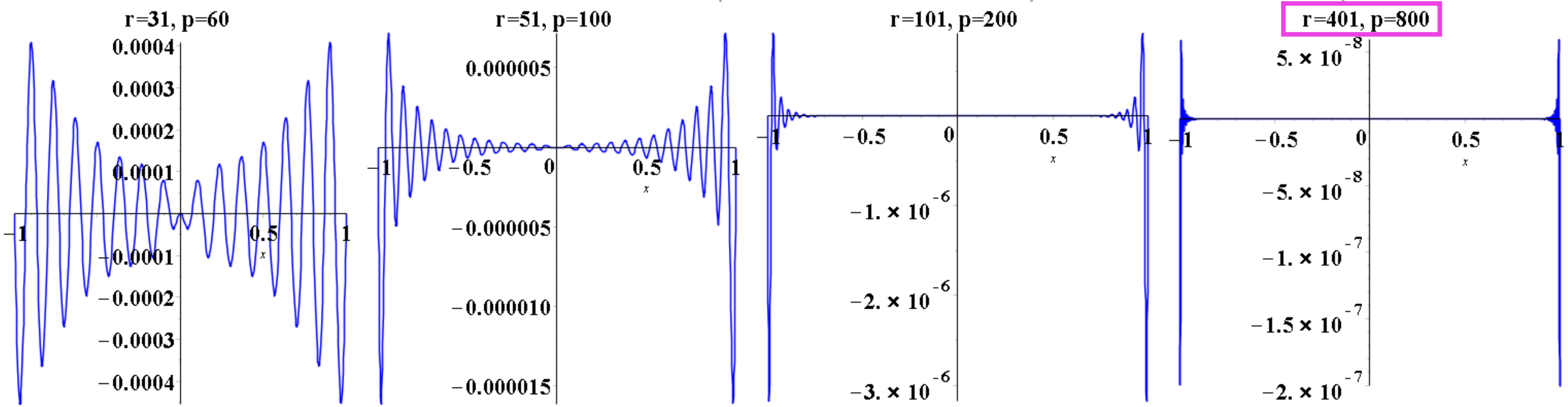
$$[x0, x0+h, x0+2h, x0+3h, x0+4h]$$

$$[y0, y1, y2, y3, y4]$$

$$\frac{\frac{1}{24}\frac{y0}{h^4(x-x0)} - \frac{1}{6}\frac{y1}{h^4(x-x0-h)} + \frac{1}{4}\frac{y2}{h^4(x-x0-2h)} - \frac{1}{6}\frac{y3}{h^4(x-x0-3h)} + \frac{1}{24}\frac{y4}{h^4(x-x0-4h)}}{\frac{1}{24h^4(x-x0)} - \frac{1}{6h^4(x-x0-h)} + \frac{1}{4h^4(x-x0-2h)} - \frac{1}{6h^4(x-x0-3h)} + \frac{1}{24h^4(x-x0-4h)}}$$

- Barycentric Reprezentation of Interpolating Polynomials
- Trefethen – *Chebfun*
- Holladay's Theorem and Optimal Splines





Splines minimize curvature & energy
Splines are curvature minimizing curves

I intend to speak about the role of Q and BQ polynomials in speedup of spline computation. But of course not only.
First of all, quartic means that the polynomial is of degree four, biquartic means that the product xy is of degree four.

To understand why are we concerned with BQP when working with BCS, we have to understand the interrelation between quartic and cubic polynomials, so
after a short motivation I present the basic idea and its consequences.
The road to our main topic leads through such topics as spline curves and surfaces, reduced tridiagonal systems or explicit splines that were triggered by the basic concept.

To the areas of using smooth curves and surfaces beyond data processing and CAD systems belong ...
They need stable, but f.o.all fast & simple techniques to be able to work online & implement them in non standard and complex environment.

Consider an interval with an arbitrary inner point u1, see the scheme.

The bicubic spline S does not approximate optimally even the closest general biquartic polynomial.

https://ludwig.guru/
LU decomposition
Environment
we are dealing with
we are concerned/troubled with
The road to ... leads through
Let's move on
Let's move to the next topic
We need not be
To our surprise/astonishment
It was an unexpected

requirements/conditions are met/fulfilled

has 2 consequences
look for/search/seek after
investigate, explore, study
investigation is going on
The investigation/story continues
to clarify - tisztazni, objasnit
to explain
there are two explanations

We did not succeed
We failed
it appeared
We left/abandoned
final result