# Finding the spectral characteristics for systems with control

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# Contents

## Goal

The study of the applicability of the method of harmonic linearization to the model with control.

## Tasks

- The initial model linearization.
- Reduction to the form necessary for harmonic linearization method application.
- Selection of criteria to be used in the future.
- Construction of equations for stability criteria and their investigation.

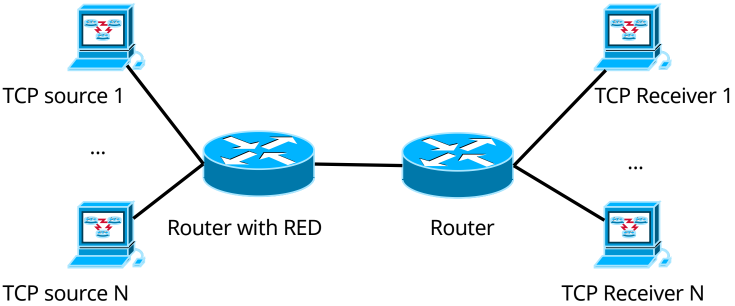


Figure 1. Dumbbell topology

# Continuous Model I

$$
\begin{cases}
\dot{W}(t) = \dfrac{1}{T(Q,t)} - \dfrac{W(t)W(t-T(Q,t))}{2T(t-T(Q,t))}p(t-T(Q,t)); \\
\dot{Q}(t) = \dfrac{W(t)}{T(Q,t)}N(t) - C; \\
\dot{\hat{Q}}(t) = -w_q C\hat{Q}(t) + w_q C Q(t).
\end{cases}
$$

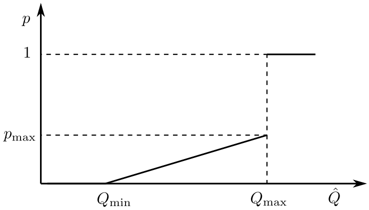


Figure 2. RED packet drop function

$$p(\hat{Q}) = \begin{cases} 0, & 0 < \hat{Q} \leqslant Q_{\min}, \\ \dfrac{\hat{Q} - Q_{\min}}{Q_{\max} - Q_{\min}} p_{\max}, & Q_{\min} < \hat{Q} \leqslant Q_{\max}, \\ 1, & \hat{Q} > Q_{\max}. \end{cases}$$

# TCP Global Synchronization I

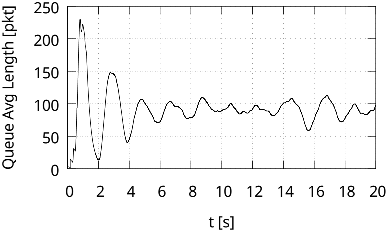


Figure 3. Dynamics of queue length change withoutwith global synchronization

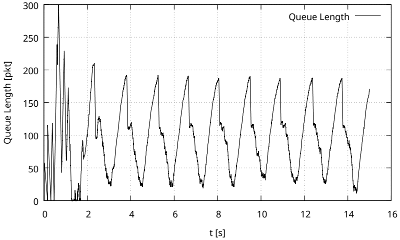


Figure 4. Dynamics of queue length change with global synchronization

# Linearization I

**The linearization of a function $f(x)$ at a point $x = a$**

$$f(x) = f(a) + \left.\frac{\partial f(x)}{\partial x}\right|_a (x - a) + \mathcal{O}\big(x^2\big).$$

# Linearization II

## Oscillatory Regime

Linearization destroys the oscillatory regime.

Harmonic linearization leaves the structure of the oscillatory regime.

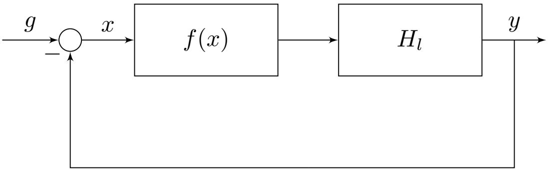


Figure 5. Block structure of the system for the harmonic linearization method

## The transfer function of the non-linear link

$$f(x) = \left[\varkappa(A) + \frac{\varkappa'(A)}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\right]x = H_{nl}(A, \partial_t)x,$$

## The harmonic linearization coefficients

$$\varkappa(A) = \frac{a_1}{A} = \frac{1}{A\pi}\int_0^{2\pi} f(A\sin(\omega t))\sin(\omega t)\,\mathrm{d}(\omega t)\,;$$

$$\varkappa'(A) = \frac{b_1}{A} = \frac{1}{A\pi}\int_0^{2\pi} f(A\sin(\omega t))\cos(\omega t)\,\mathrm{d}(\omega t)\,.$$

# The Nyquist–Mikhailov Criterion I

## The system characteristic function

$$1 + H_o(\mathrm{i}\omega) = 0,$$
$$H_o(\mathrm{i}\omega) := H_l(\mathrm{i}\omega) H_{nl}(A, \mathrm{i}\omega).$$

$H_o$ — the transfer function of the open-loop system.

# The Nyquist–Mikhailov Criterion II

## Goldfarb's hodograph

$$H_l(\mathrm{i}\omega) = -\frac{1}{\varkappa(A) + \mathrm{i}\varkappa'(A)}.$$

# The Nyquist–Mikhailov Criterion III

## Cochenburger's hodograph

$$\varkappa(A) + \mathrm{i}\varkappa'(A) = -\frac{1}{H_l(\mathrm{i}\omega)}.$$

## Harmonic linearization algorithm

- Finding equilibrium point;
- Constructing constraints equations;
- Varying the right part of the system;
- Linearization of the system;
- Linearization of the drop function;
- Constructing a block representation of the linearized model RED.

## Equilibrium point

$$\begin{cases} 0 = \dfrac{1}{T_f} - \dfrac{W_f^2}{2T_f} p_f; \\ 0 = \dfrac{W_f}{T_f} N_f - C; \\ 0 = -w_q C \hat{Q}_f + w_q C Q_f. \end{cases}$$

## Constraints equations

$$
\begin{cases}
p_f = \dfrac{2}{W_f^2}; \\
W_f = \dfrac{CT_f}{N_f}; \\
\hat{Q}_f = Q_f.
\end{cases}
$$

## Varying the right part of the system

$$\begin{cases} L_W(W, W_T, Q, p) = \dfrac{1}{T} - \dfrac{W W_T}{2T} p; \\ L_Q(W, Q) = \dfrac{W}{T} N - C; \\ L_{\hat{Q}}(\hat{Q}, Q) = -w_q C \hat{Q} + w_q C Q. \end{cases}$$

Variables: $W := W(t)$, $W_T := W(t - T)$, $Q := Q(t)$, $p := p(t - T)$.

## Linearized system

$$\begin{cases}
\delta W(s) = -\dfrac{1}{s + \frac{N}{CT_f^2}\left(1 + \mathrm{e}^{-sT_f}\right)} \dfrac{C^2 T_f}{2N^2} \mathrm{e}^{-sT_f}\, \delta p(s)\,; \\
\delta Q(s) = \dfrac{1}{s + \frac{1}{T_f}} \dfrac{N}{T_f}\, \delta W(s)\,; \\
\delta \hat{Q}(s) = \dfrac{1}{1 + \frac{s}{w_q C}}\, \delta Q(s)\,.
\end{cases}$$

## Linearized drop function

$$\delta p(s) = P_{\mathrm{RED}} \frac{1}{1 + \frac{s}{w_q C}} \delta Q(s) \,.$$

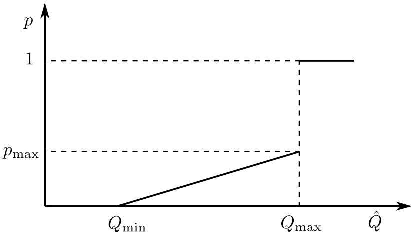


Figure 6. The function $p$

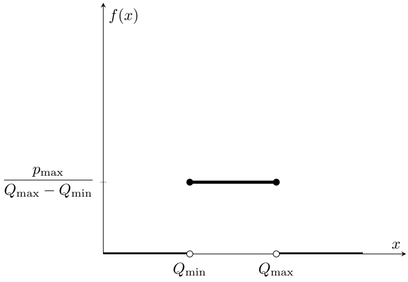


Figure 7. The function $P_{\mathrm{RED}}$

## Block representation of the linearized RED model for harmonic linearization

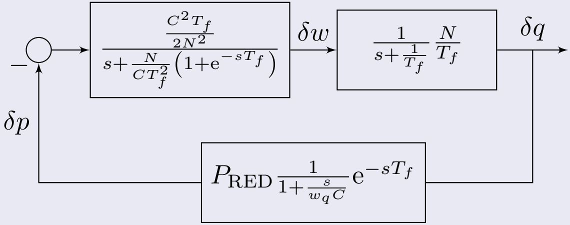


Figure 8. Block representation of the linearized RED model

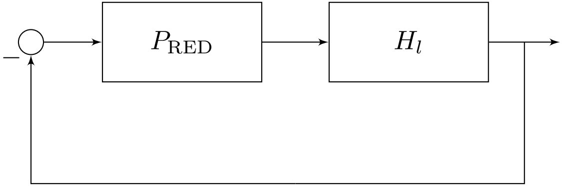


Figure 9. Block representation of the linearized RED model for harmonic linearization

## The Nyquist equation (Goldfarb's hodograph)

$$\frac{1}{\mathrm{i}\omega + \frac{N}{CT_f^2}\left(1 + \mathrm{e}^{-\mathrm{i}\omega T_f}\right)} \frac{1}{\mathrm{i}\omega + \frac{1}{T_f}} \frac{1}{1 + \frac{\mathrm{i}\omega}{w_q C}} \frac{C^2}{2N} \mathrm{e}^{-\mathrm{i}\omega T_f} =$$

$$= -\frac{A\pi}{4p_{\max}} \left[ \frac{1}{Q_{\max} - Q_{\min}} \left( \sqrt{1 - \frac{Q_{\min}^2}{A^2}} - \sqrt{1 - \frac{Q_{\max}^2}{A^2}} \right) + \mathrm{i}\frac{1}{A} \right]^{-1}.$$

## The Nyquist equation (Cochenburger's hodograph)

$$\mathrm{i}\omega + \frac{N}{CT_f^2}\left(1 + \mathrm{e}^{-\mathrm{i}\omega T_f}\right)\left(\mathrm{i}\omega + \frac{1}{T_f}\right)\left(1 + \frac{\mathrm{i}\omega}{w_q C}\right)\frac{2N}{C^2}\mathrm{e}^{\mathrm{i}\omega T_f} =$$
$$= \frac{4p_{\max}}{A\pi}\left[\frac{1}{Q_{\max} - Q_{\min}}\left(\sqrt{1 - \frac{Q_{\min}^2}{A^2}} - \sqrt{1 - \frac{Q_{\max}^2}{A^2}}\right) + \mathrm{i}\frac{1}{A}\right].$$

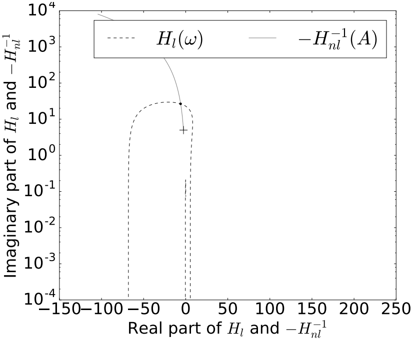


Figure 10. Nyquist plot for system (Goldfarb's hodograph)

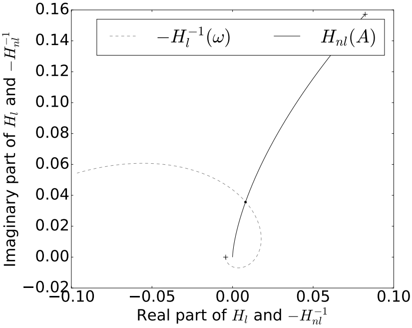


Figure 11. Nyquist plot for system (Cochenburger's hodograph)

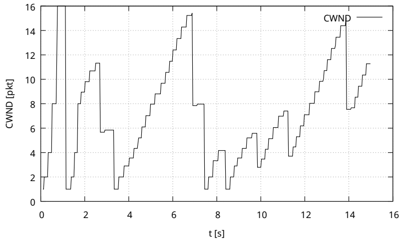


Figure 12. Change the size of the sliding window on the source

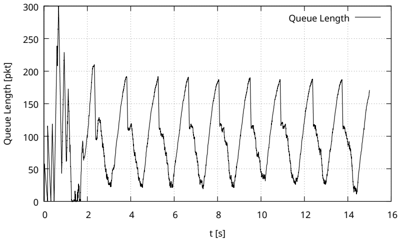


Figure 13. Queuing on the router with RED

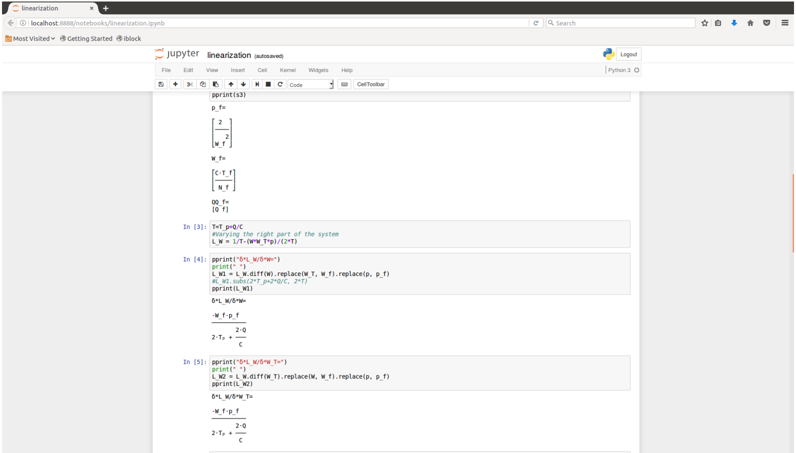


Figure 14. Automation of calculations using Jupyter Notebook

7 Notation

8 Harmonic Linearization

We use the following notation:

- $W$ — the average TCP window size;
- $Q$ — the average queue size;
- $\hat{Q}$ — the exponentially weighted moving average (EWMA) of the queue size average;
- $C$ — the queue service intensity;
- $T$ — full round-trip time; $T = T_p + \frac{Q}{C}$, where $T_p$ — free network round-trip time (excluding delays in hardware); $\frac{Q}{C}$ — the time spent by a packet in the queue;
- $N$ — number of TCP sessions;
- $p$ — packet drop function.

The input of a nonlinear element is fed by free harmonic oscillations:

$$x(t) = A\sin(\omega t).$$

At the output of the nonlinear element $f(x)$ we obtain a periodic signal. We expand it into a Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \sin(k\omega t) + b_k \cos(k\omega t)),$$

where the coefficients of the Fourier series have the following form:

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(A\sin(\omega t))\sin(k\omega t)\,\mathrm{d}(\omega t)\,;$$
$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(A\sin(\omega t))\cos(k\omega t)\,\mathrm{d}(\omega t)\,;\quad k = \overline{1,\infty}.$$