



Introduction to hydrodynamic description of Heavy-Ion Collisions

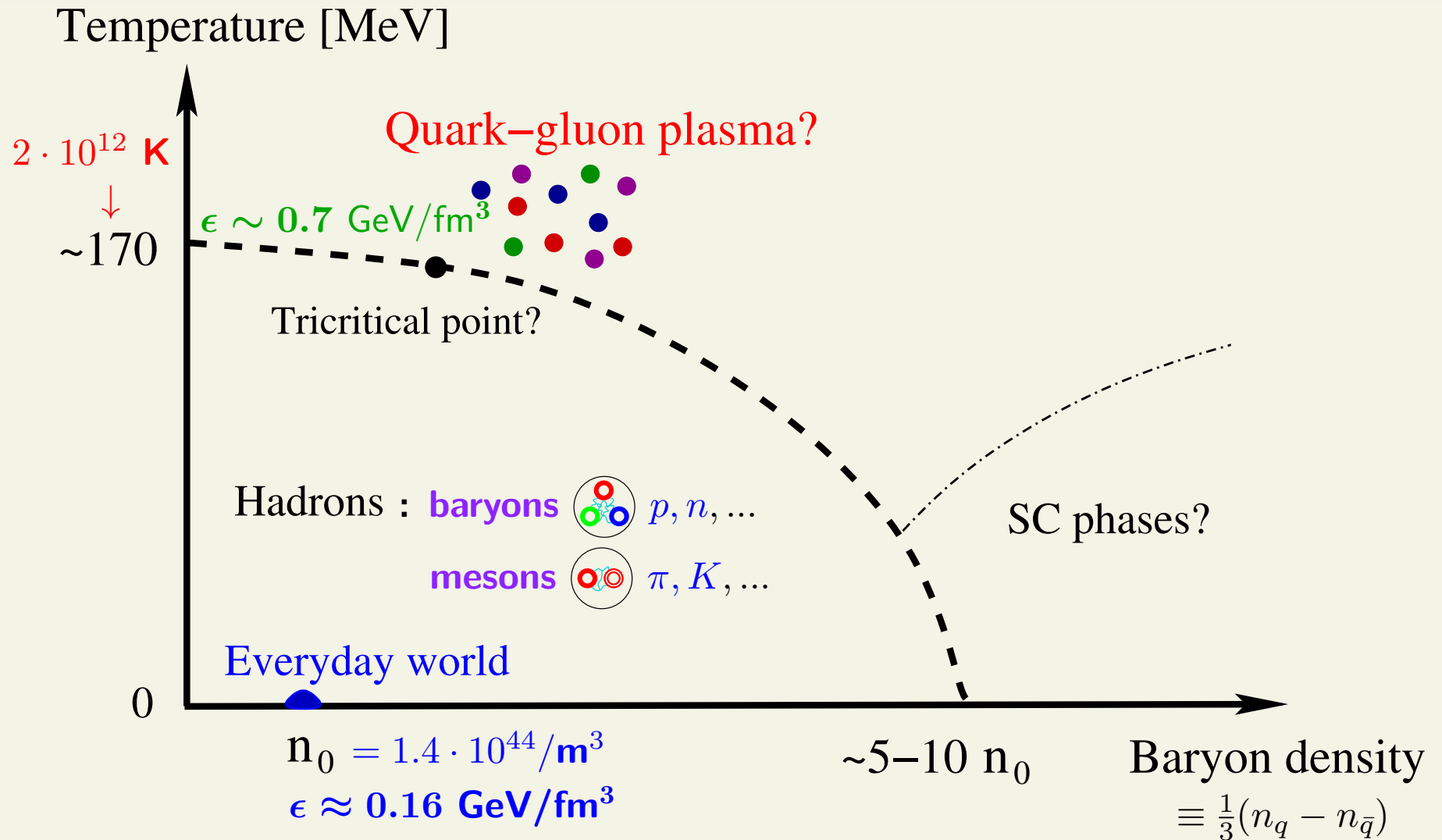
Pasi Huovinen
Uniwersytet Wrocławski

Helmholz International Summer School
**“Matter under Extreme Conditions in Heavy-Ion Collisions
and Astrophysics”**

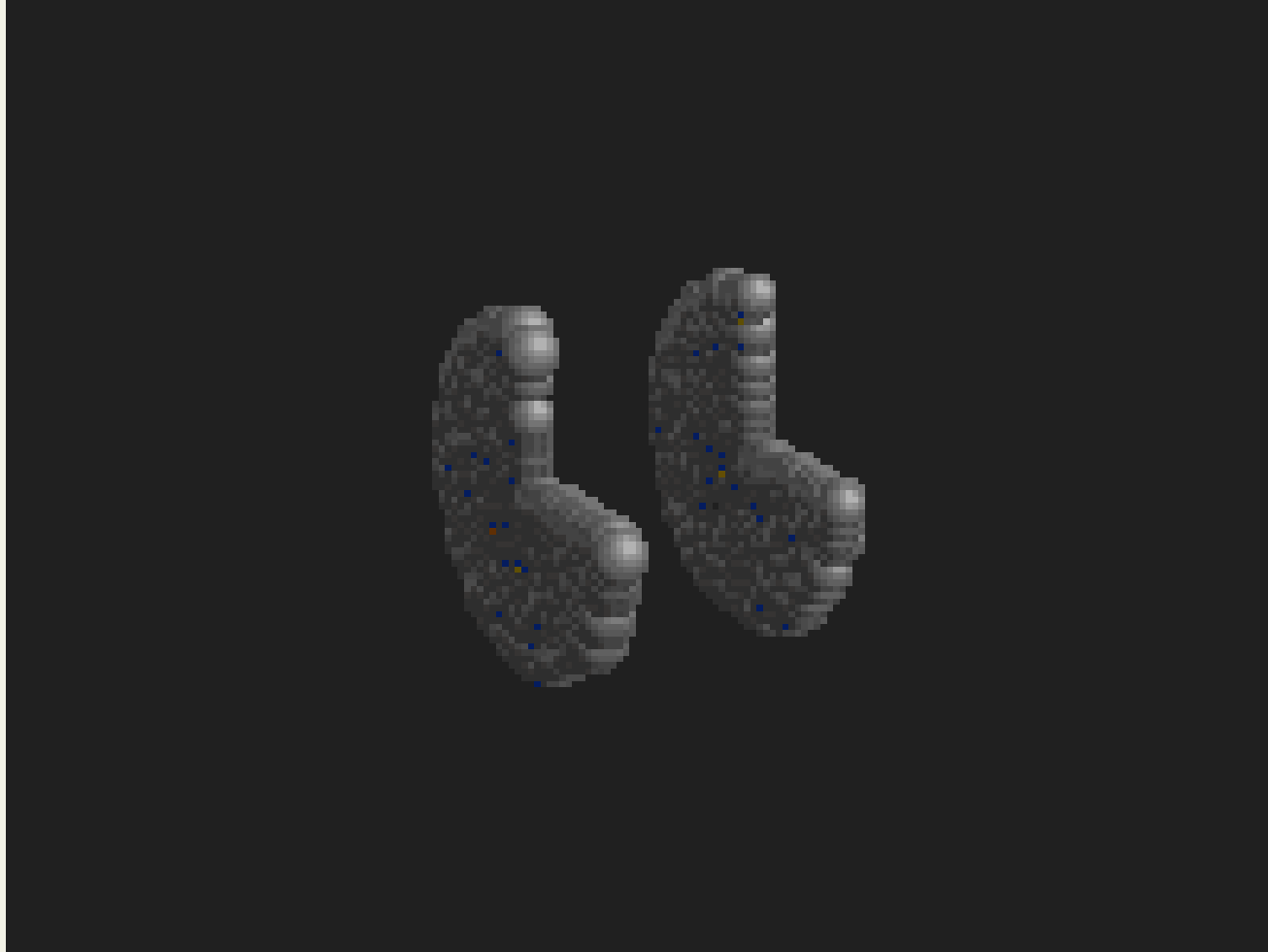
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Nuclear phase diagram



Heavy-ion collision



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Heavy-ion collision



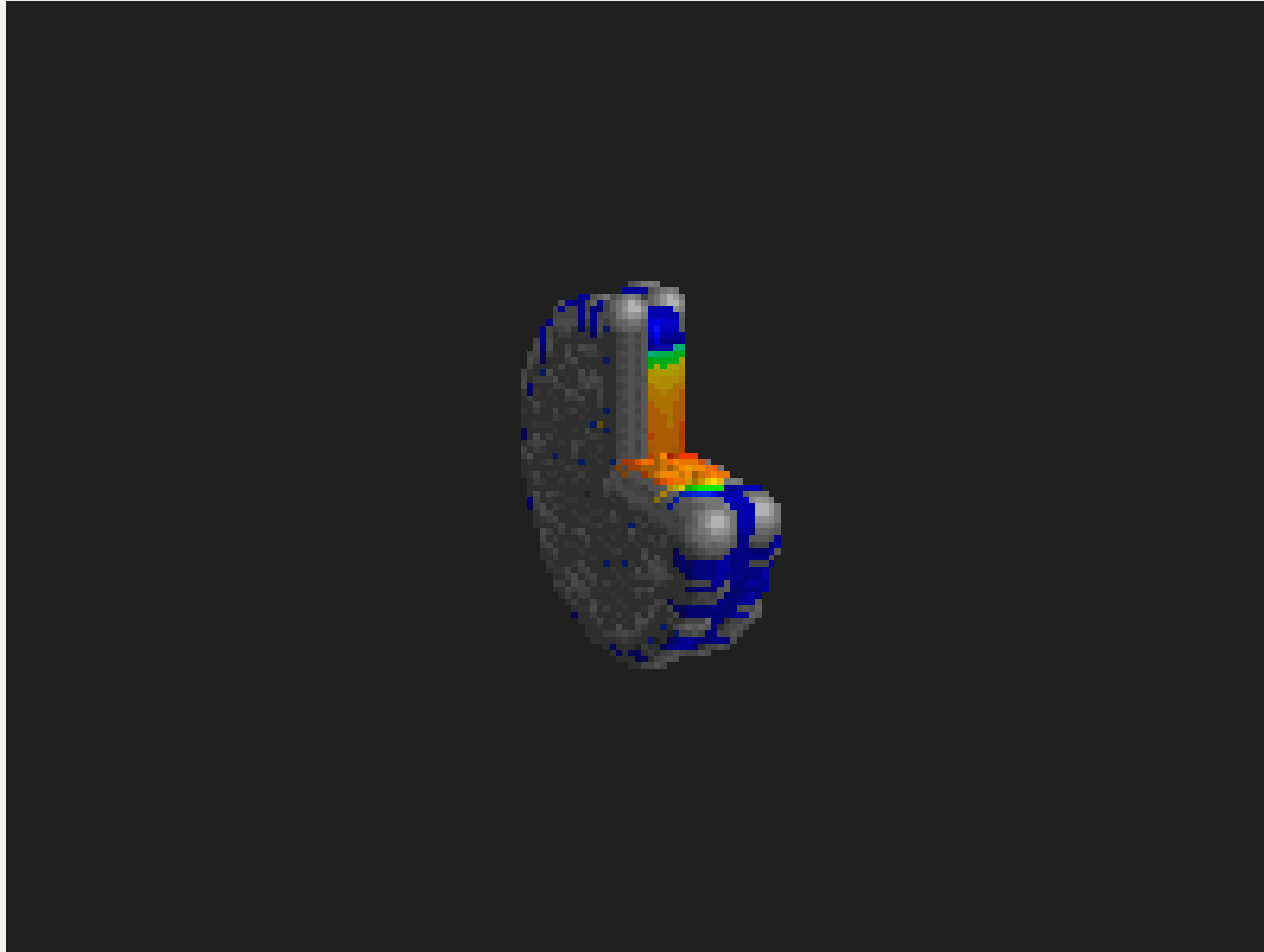
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Heavy-ion collision



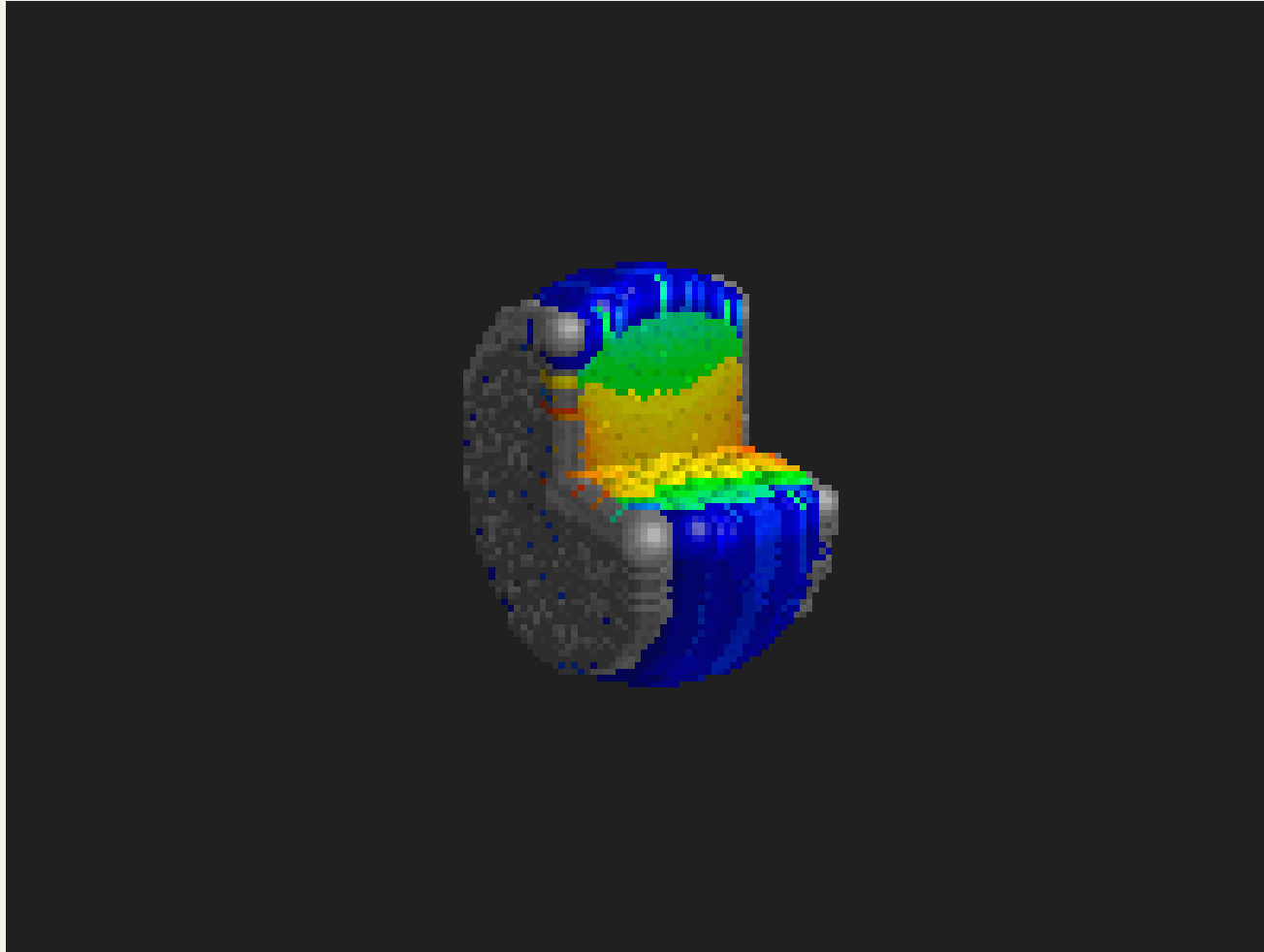
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Heavy-ion collision



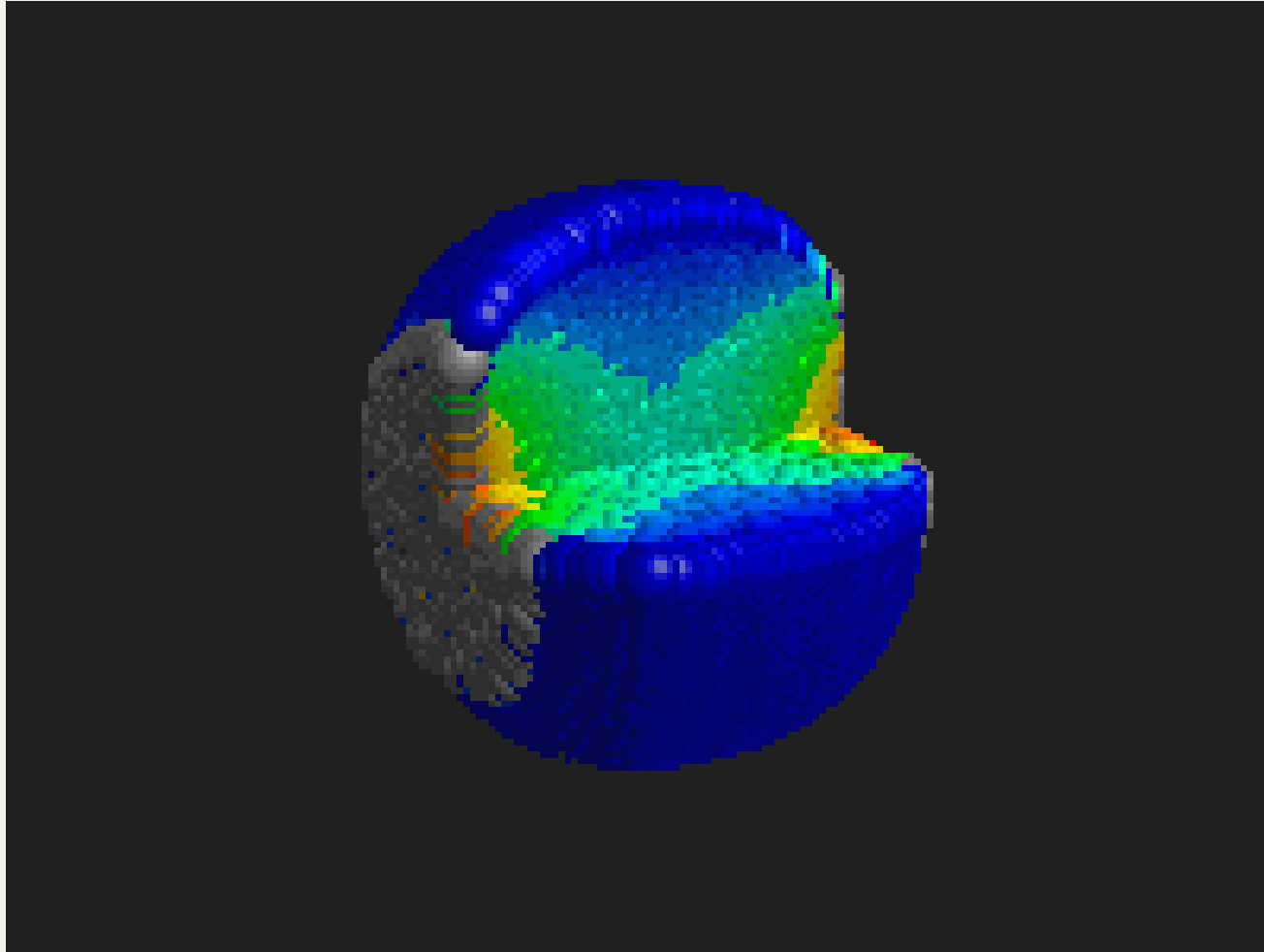
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Heavy-ion collision



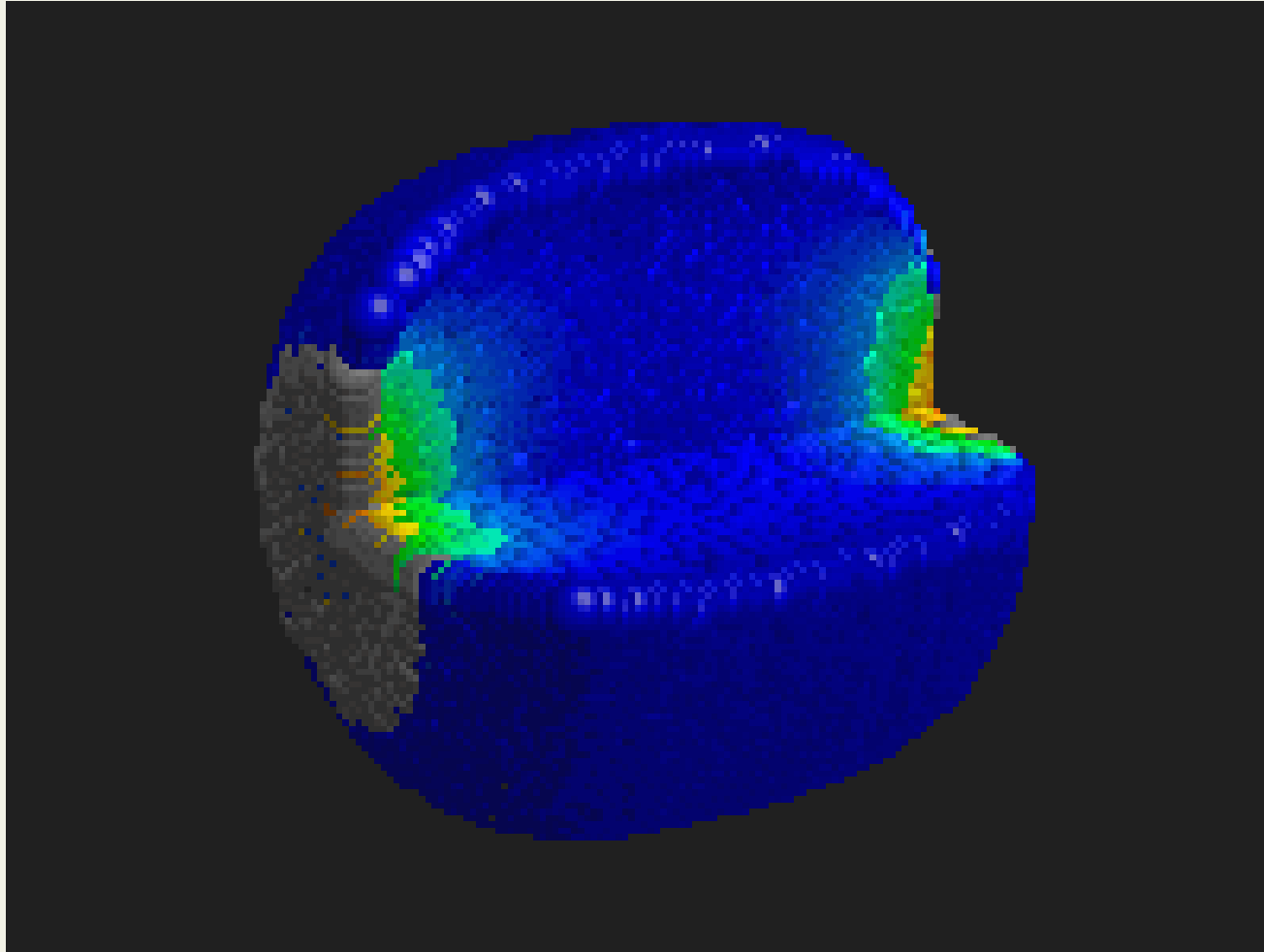
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Heavy-ion collision

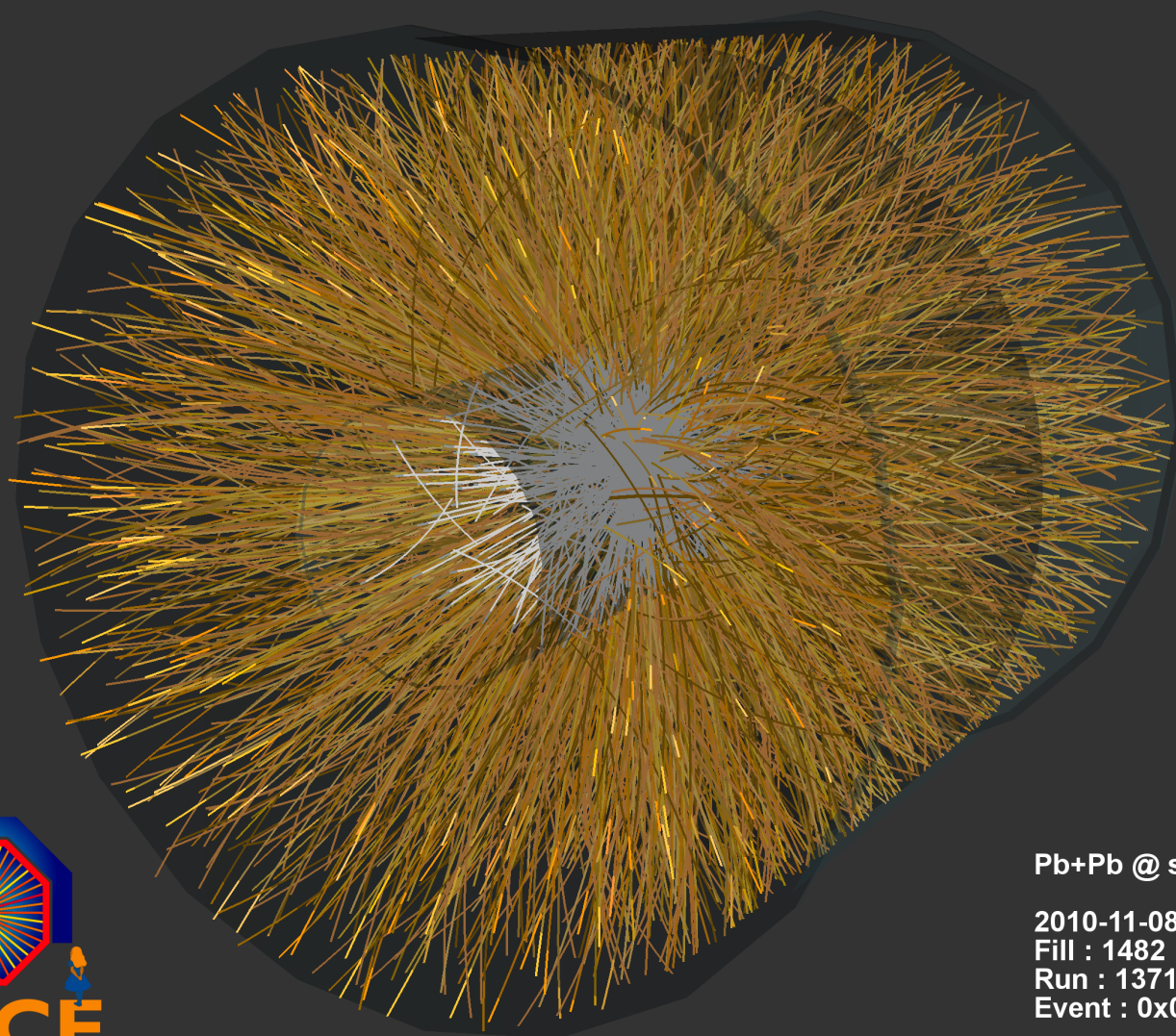


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Heavy-ion collision



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Pb+Pb @ $\sqrt{s} = 2.76$ ATeV

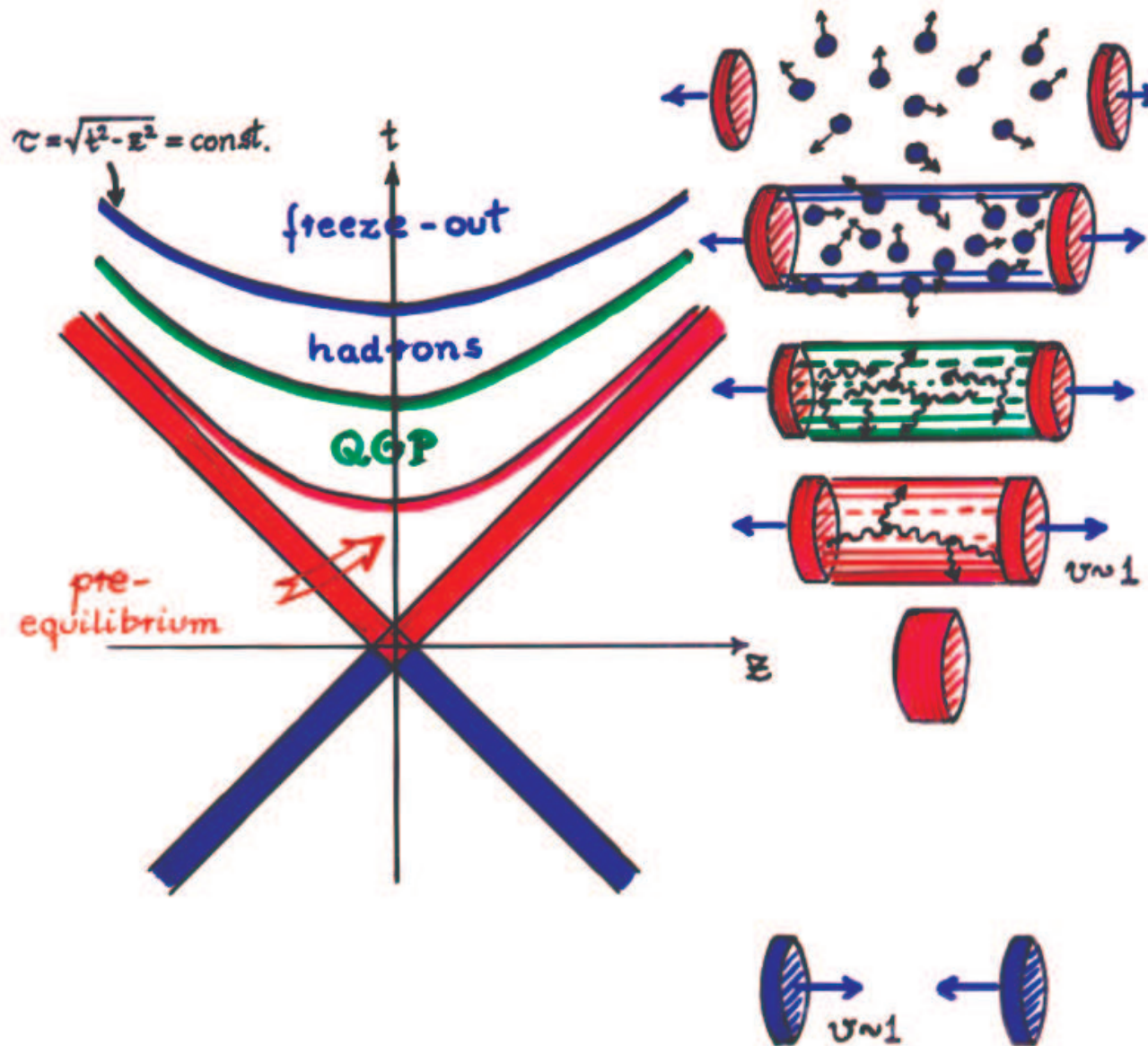
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The space-time picture:



Transient matter

- lifetime
 $t \sim 10 \text{ fm}/c$
 $\sim 10^{-23}$ seconds
- small size
 $r \sim 10 \text{ fm}$
 $\sim 10^{-14} \text{ m}$
- rapid expansion

Multiplicity @ LHC

~ 15000

Conservation laws

Conservation of energy and momentum:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

Conservation of charge:

$$\partial_\mu N^\mu(x) = 0$$

Local conservation of particle number and energy-momentum.

$\Longleftrightarrow$ **Hydrodynamics!**

This can be generalized to **multicomponent systems** and **systems with several conserved charges**:

$$\partial_\mu N_i^\mu = 0,$$

$i =$ **baryon number**, **strangeness**, **charge** . . .

Consider only **baryon number conservation**, $i = B$.

⇒ **5 equations** contain **14 unknowns**!

⇒ **The system of equations does not close.**

⇒ **Provide 9 additional equations** or
Eliminate 9 unknowns.

So **what are the components** of $T^{\mu\nu}$ and N^μ ?

- N^μ and $T^{\mu\nu}$ can be decomposed with respect to **arbitrary, normalized, time-like 4-vector** u^μ ,

$$u_\mu u^\mu = 1$$

- Define a **projection operator**

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \quad \Delta^{\mu\nu} u_\nu = 0,$$

which **projects on the 3-space** orthogonal to u^μ .

- Then

$$N^\mu = nu^\mu + \nu^\mu$$

where

$n = N^\mu u_\mu$ is (baryon) charge density in the frame where $u = (1, 0)$, local rest frame, LRF

$\nu^\mu = \Delta^{\mu\nu} N_\nu$ is charge flow in LRF,

and

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}$$

$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu$ energy density in LRF

$P \equiv -\frac{1}{3} \Delta^{\mu\nu} T_{\mu\nu}$ isotropic pressure in LRF

$q^\mu \equiv \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$ energy flow in LRF

$\pi^{\mu\nu} \equiv [\frac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\beta \Delta^\mu_\alpha) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta}] T^{\alpha\beta}$
(trace-free) stress tensor in LRF

- The 14 unknowns in 5 equations:

$$\left. \begin{array}{cc} N^\mu & 4 \\ T^{\mu\nu} & 10 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{cc} n, \epsilon, P & 3 \\ q^\mu & 3 \\ \nu^\mu & 3 \\ \pi^{\mu\nu} & 5 \end{array} \right.$$

- So far u^μ is **arbitrary**. It attains a **physical meaning** by relating it to N^μ or $T^{\mu\nu}$:

1. **Eckart frame:**

$$u_E^\mu \equiv \frac{N^\mu}{\sqrt{N_\nu N^\nu}}$$

u^μ is 4-velocity of charge flow, $\nu^\mu = 0$.

The 14 unknowns are $n, \epsilon, P, q^\mu, \pi^{\mu\nu}, u^\mu$.

2. **Landau frame:**

$$u_L^\mu \equiv \frac{T^{\mu\nu} u_\nu}{\sqrt{u_\alpha T^{\alpha\beta} T_{\beta\gamma} u^\gamma}}$$

u^μ is 4-velocity of energy flow, $q^\mu = 0$.

The 14 unknowns are $n, \epsilon, P, \nu^\mu, \pi^{\mu\nu}, u^\mu$.

- In general, the hydrodynamical equations are **not closed** and **cannot be solved uniquely**.

Ideal hydrodynamics

Suppose particles are in **local thermodynamical equilibrium**, i.e., single particle phase space distribution function is given by:

$$f_i(x, k) = \frac{g}{(2\pi)^3} \left[\exp \left(\frac{k_\mu u^\mu(x) - \mu(x)}{T(x)} \right) \pm 1 \right]^{-1}$$

where

$T(x)$ and $\mu(x)$: **local temperature and chemical potential**
 $u(x)^\mu$: **local 4-velocity of fluid flow.**

Then kinetic theory definitions give

$$N^\mu(x) \equiv \sum_i q_i \int \frac{d^3\mathbf{k}}{E} k^\mu f_i(x, k) = n(T, \mu) u^\mu$$

$$\begin{aligned} T^{\mu\nu}(x) &\equiv \sum_i \int \frac{d^3\mathbf{k}}{E} k^\mu k^\nu f_i(x, k) \\ &= (\epsilon(T, \mu) + P(T, \mu)) u^\mu u^\nu - P(T, \mu) g^{\mu\nu} \end{aligned}$$

where

$$n(T, \mu) = \sum_i q_i \int d^3\mathbf{k} f_i(x, E) \text{ is local charge density,}$$

$$\epsilon(T, \mu) = \sum_i \int d^3\mathbf{k} E f_i(x, E) \text{ is local energy density and}$$

$$P(T, \mu) = \sum_i \int d^3\mathbf{k} \frac{\mathbf{k}^2}{3E} f_i(x, E) \text{ is local pressure.}$$

Note! $f(x, E)$ is distribution in **local rest frame**: $u^\mu = (1, 0)$.

→ **Local** thermodynamical **equilibrium** implies **there is no viscosity**:

$$\nu^\mu = q^\mu = \pi^{\mu\nu} = 0.$$

Ideal fluid approximation:

$$N^\mu = nu^\mu$$

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}$$

- **Local equilibrium** $\Rightarrow$ **no viscosity**: $\nu^\mu = q^\mu = \pi^{\mu\nu} = 0$.
- Now N^μ and $T^{\mu\nu}$ contain **6 unknowns**, ϵ , P , n and u^μ , but there are still only **5 equations**!
- In thermodynamical equilibrium ϵ , P and n are not independent! They are specified by two variables, T and μ .
- The **equation of state** (EoS), $P(T, \mu)$ eliminates one unknown!
- Any **equation of state** of the form

$$P = P(\epsilon, n)$$

closes the system of hydrodynamic equations and makes it **uniquely solvable** (given initial conditions).

Remark: $P = P(\epsilon, n)$ is not a **complete equation of state** in a thermodynamical sense.

A complete equation of state allows to compute all thermodynamic variables.

For example, $s = s(\epsilon, n)$: $ds = 1/T d\epsilon - \mu/T dn$ (1st law of thermod.)

$$\frac{1}{T} = \left. \frac{\partial s}{\partial \epsilon} \right|_n, \quad \frac{\mu}{T} = - \left. \frac{\partial s}{\partial n} \right|_\epsilon, \quad P = Ts + \mu n - \epsilon$$

$P = P(\epsilon, n)$ **does not work!**

$$\left. \frac{\partial P}{\partial \epsilon} \right|_n = ? \quad \left. \frac{\partial P}{\partial n} \right|_\epsilon = ?$$

However, $P = P(T, \mu)$ **does work!**

$$dP = s dT + n d\mu \quad \Rightarrow \quad s = \left. \frac{\partial P}{\partial T} \right|_\mu, \quad n = \left. \frac{\partial P}{\partial \mu} \right|_T$$

Entropy in ideal fluid

is conserved!

$$\partial_\mu S^\mu = 0$$

where $S^\mu = su^\mu$.

Equations of motion

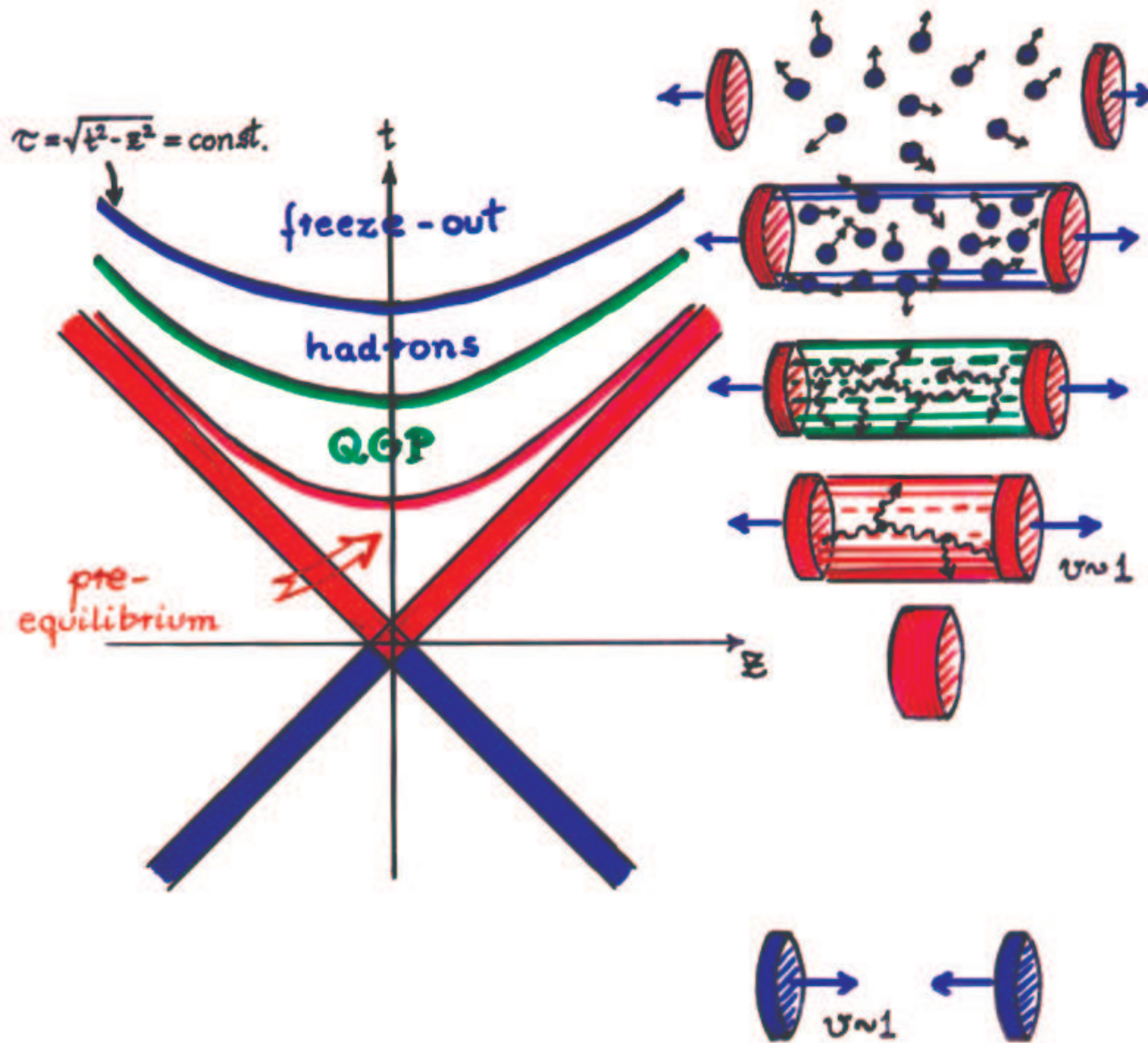
Conservation laws lead to the **equations of motion** for relativistic fluid:

$$\begin{aligned} Dn &= -n\partial_\mu u^\mu \\ D\epsilon &= -(\epsilon + P)\partial_\mu u^\mu \\ (\epsilon + P)Du^\mu &= \nabla^\mu P, \end{aligned}$$

where

$$D = u^\mu \partial_\mu \quad \text{and} \quad \nabla^\mu = \Delta^{\mu\nu} \partial_\nu.$$

The space-time picture:



Usefulness of hydro?

- Initial state: unknown
 - Equation of state: unknown
 - Transport coefficients: unknown
 - Freeze-out: unknown
- } $\Rightarrow$ Predictive power?

– “*Hydro doesn’t know where to start nor where to end*” (M. Prakash)

Usefulness of hydro?

- Initial state: **unknown**
 - Equation of state: **want to study**
 - Transport coefficients: **want to study**
 - Freeze-out: **unknown**
- } $\Rightarrow$ **Predictive power?**

$\Rightarrow$ **Need More Constraints!**

“Hydrodynamical method”

1. Use **another model** to fix unknowns (and **add new assumptions. . .**)
 - **initial:** color glass condensate or pQCD+saturation
 - **initial and/or final:** hadronic cascade
 - etc.
2. Use data to fix parameters:

Principle

- use one set of data

$\Longleftrightarrow$

Example @ RHIC

$$\left. \frac{dN}{dy p_T dp_T} \right|_{b=0} \quad \text{and} \quad \frac{dN}{dy}(b)$$

- fix parameters to fit it

$\Longleftrightarrow$

$$\begin{cases} \epsilon_{0,\max} = 29.6 \text{ GeV/fm}^3 \\ \tau_0 = 0.6 \text{ fm}/c \\ T_{fo} = 130 \text{ MeV} \end{cases}$$

- predict another set of data

$\Longleftrightarrow$

HBT, photons & dileptons,
elliptic flow. . .

Equations of motion

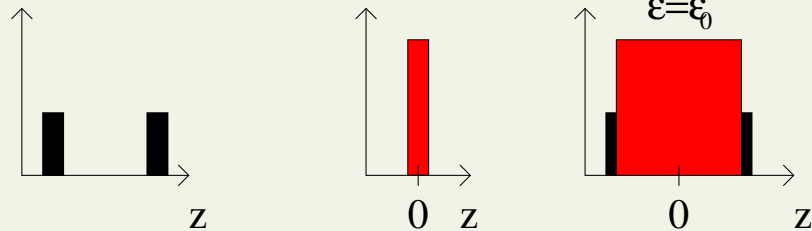
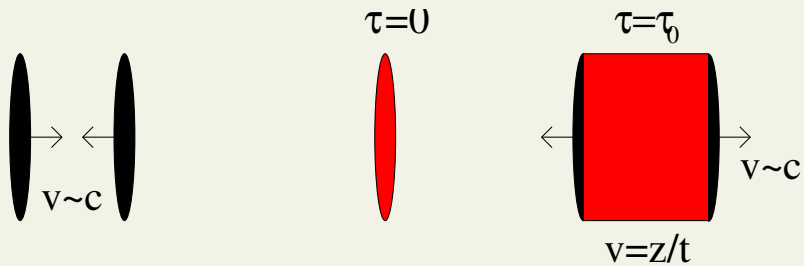
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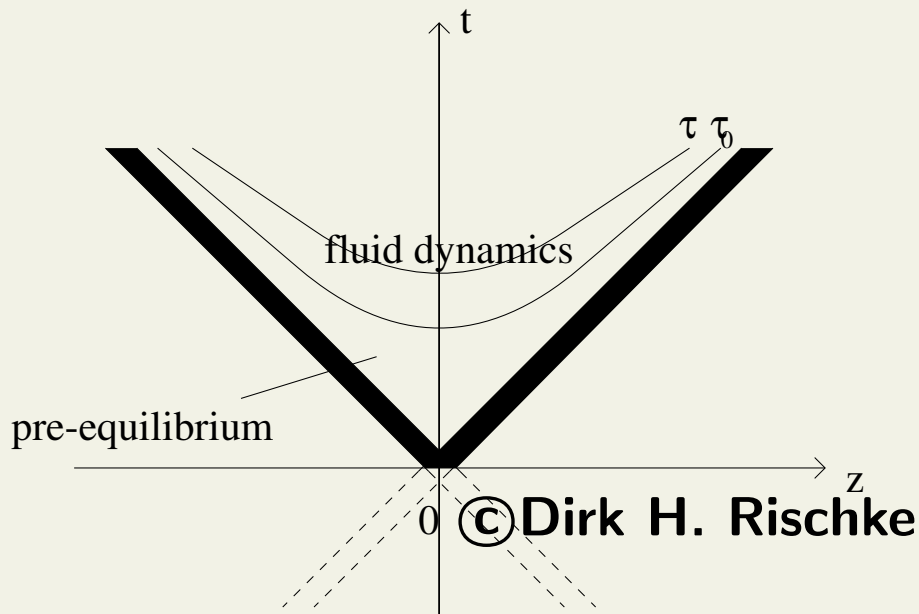
where

$$D = u^\mu \partial_\mu \quad \text{and} \quad \nabla^\mu = \Delta^{\mu\nu} \partial_\nu.$$

Bjorken hydrodynamics

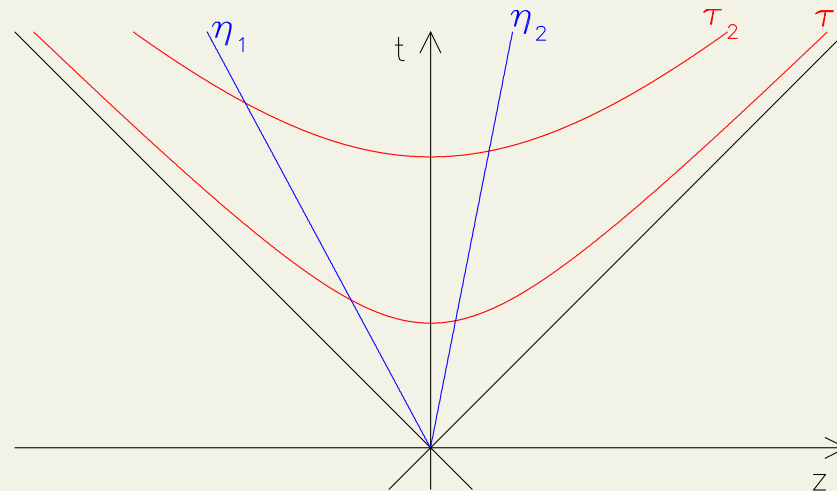


- At very large energies, $\gamma \rightarrow \infty$ and “Landau thickness” $\rightarrow 0$
- Lack of longitudinal scale $\Rightarrow$ **scaling flow**



$$v = \frac{z}{t}$$

- Practical coordinates to describe scaling flow expansion are



- Longitudinal proper time τ :

$$\tau \equiv \sqrt{t^2 - z^2} \quad \Leftrightarrow \quad t = \tau \cosh \eta$$

- Space-time rapidity η :

$$\eta = \frac{1}{2} \ln \frac{t+z}{t-z} \quad \Leftrightarrow \quad z = \tau \sinh \eta$$

- **Scaling flow** $v = z/t \Rightarrow$ **fluid flow rapidity** $y = \eta$:

$$y = \frac{1}{2} \ln \frac{1+v}{1-v} = \frac{1}{2} \ln \frac{1+z/t}{1-z/t} = \eta$$

- **Ignore transverse expansion:**

Hydrodynamic equations turn out to be **particularly simple**:

$$\left. \frac{\partial \epsilon}{\partial \tau} \right|_{\eta} = -\frac{\epsilon + P}{\tau} \quad (1)$$

$$\left. \frac{\partial P}{\partial \eta} \right|_{\tau} = 0 \quad (2)$$

$$\left. \frac{\partial n}{\partial \tau} \right|_{\eta} = -\frac{n}{\tau} \quad (3)$$

- **Eq. (2) $\Rightarrow$**

- **No force between fluid elements with different η !**
- $P = P(\tau)$, **no η -dependence!**

- Eq. (2) + thermodynamics:

$$0 = \left. \frac{\partial P}{\partial \eta} \right|_{\tau} = s \left. \frac{\partial T}{\partial \eta} \right|_{\tau} + n \left. \frac{\partial \mu}{\partial \eta} \right|_{\tau}$$

If $n = 0$, $T = T(\tau) \Rightarrow T = \text{const. on } \tau = \text{const. surface.}$

- In general T and ϵ not constant on $\tau = \text{const. surface}$, but usually they are assumed to be
 $\Rightarrow$ **boost invariance: the system looks the same in all reference frames!**

$$\epsilon = \epsilon(\tau), \quad n = n(\tau)$$

- **Note that still**

$$\frac{\partial}{\partial \eta} T^{\mu\nu} \neq 0 \neq \frac{\partial}{\partial \eta} u^{\mu}$$

Vector and tensor quantities at finite η Lorentz boosted from values at $\eta = 0$

- Thermodynamics:

$$\begin{aligned}d\epsilon &= T ds + \mu dn \\ \epsilon + P &= Ts + \mu n\end{aligned}$$

- Eq. (1):

$$\frac{\partial \epsilon}{\partial \tau} + \frac{\epsilon + P}{\tau} = 0$$

$$\Rightarrow T \frac{\partial s}{\partial \tau} + \mu \frac{\partial n}{\partial \tau} + T \frac{s}{\tau} + \mu \frac{n}{\tau} = 0$$

$$\text{(Eq. (3))} \Rightarrow \frac{\partial s}{\partial \tau} + \frac{s}{\tau} = 0$$

$$\Rightarrow s(\tau) = s_0 \frac{\tau_0}{\tau}$$

$$\Rightarrow s\tau = \text{const.} \Rightarrow dS/d\eta = \text{const}$$

independent of the equation of state!

- Time evolution of baryon density:

$$\text{Eq. (3)} \Rightarrow n(\tau) = n_0 \frac{\tau_0}{\tau} \Rightarrow dN/d\eta = \text{const}$$

also independent of the EoS.

- Time evolution of energy density and temperature depend on the EoS.
- Assume ideal gas equation of state, $P = \frac{1}{3}\epsilon$, $\epsilon \propto T^4$:

$$\begin{aligned} \text{Eq. (1)} \quad &\Rightarrow \frac{\partial \epsilon}{\partial \tau} + \frac{4\epsilon}{3\tau} = 0 \\ &\Rightarrow \epsilon(\tau) = \epsilon_0 \left(\frac{\tau_0}{\tau} \right)^{\frac{4}{3}} \\ &\Rightarrow T(\tau) = T_0 \left(\frac{\tau_0}{\tau} \right)^{\frac{1}{3}} \end{aligned}$$

- Note: τ_0 : initial time, thermalization time

Application: Initial energy density estimate

1. “Bjorken estimate”

- **At** $y = 0$, $E_T = E$
- **Thus measuring**

$$\left. \frac{dE_T}{dy} \right|_{y=0}$$

gives total energy at $y = 0$.

- **Estimate the initial volume:**

$$V = A \Delta z = \pi R^2 \tau_0 \Delta \eta$$

- **Thus**

$$\epsilon = \frac{1}{\pi R^2 \tau_0 \Delta \eta} E = \frac{1}{\pi R^2 \tau_0} \frac{dE_T}{dy}$$

- Take $R = 6.3 \text{ fm}$ and $\tau_0 = 1 \text{ fm/c}$:

$$\text{@ SPS: } \frac{dE_T}{dy} \approx 400 \text{ GeV} \rightarrow \epsilon \sim 3.2 \text{ GeV/fm}^3$$

$$\text{@ RHIC: } \frac{dE_T}{dy} \approx 620 \text{ GeV} \rightarrow \epsilon \sim 5.0 \text{ GeV/fm}^3$$

- Note that in this approach

$$\epsilon(\tau) = \epsilon_0 \frac{\tau_0}{\tau}$$

No longitudinal work is done.

- Pressure **does work** during expansion, $dE = -Pd\tau$:

$$\frac{\partial \epsilon}{\partial \tau} = -\frac{\epsilon + P}{\tau} \Rightarrow d(\epsilon\tau) = -Pd\tau$$

Highly nontrivial

2. Entropy conservation

- Assume **ideal gas of massless particles**:

$$s = 4n \Rightarrow \frac{dS}{dy} = 4 \frac{dN}{dy}$$

$$s = \frac{4g}{\pi^2} T^3$$

$$\epsilon = \frac{3g}{\pi^2} T^4$$

- With $s\tau = \text{const.}$ these give

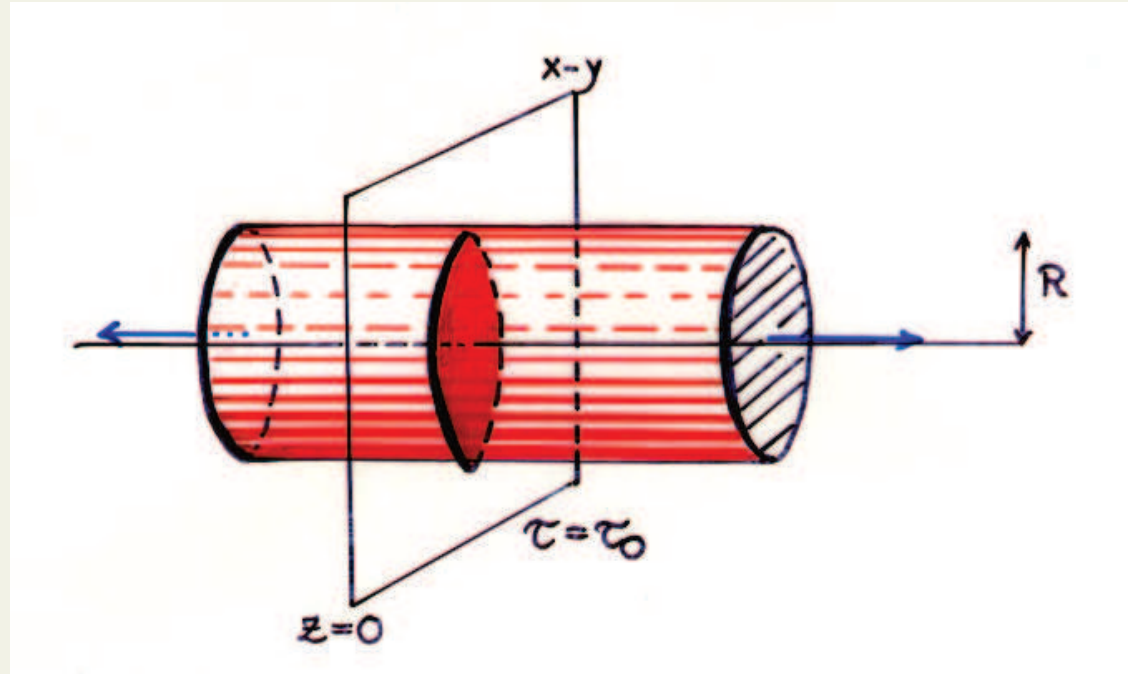
$$\epsilon_0 = \frac{3}{\pi^{\frac{2}{3}} R^{\frac{8}{3}} \tau_0^{\frac{4}{3}} g^{\frac{1}{3}}} \left(\frac{dN}{dy} \right)^{\frac{4}{3}}$$

$$\textcolor{blue}{@ RHIC:} \quad \frac{dN}{dy} \approx 1000$$

$$g = 40 \quad (\text{2 flavours} + \text{gluons})$$

$$\Rightarrow \epsilon_0 \approx \textcolor{blue}{6.0 \text{ GeV/fm}^3}$$

Transverse expansion and flow



- Transverse expansion will set in **latest** at $\tau = R/c_s \approx 10$ fm
- **Lifetimes in one dimensional expansion** ~ 30 fm
- **One dimensional** expansion an **oversimplification**
- **2+1D**: longitudinal Bjorken, transverse expansion solved numerically
- **3+1D**: expansion in all directions solved numerically

Initial conditions

- **Initial time** from early thermalization argument (+finetuning. . .)
- **Total entropy** to fit the multiplicity
- **Density distribution?**
- Multiplicity is proportional to the number of participants
- Glauber model: number of participants/binary collisions

$$N_{part}(b) = \int dx dy T_A(x + b/2, y) [\dots$$

where

$$T_A(x, y) = \int_{-\infty}^{\infty} dz \rho(x, y, z) \quad \text{and} \quad \rho(x, y, z) = \frac{\rho_0}{1 + e^{\frac{r-R_0}{a}}}$$

are nuclear thickness function and nuclear density distribution

- “Differential Optical Glauber:”

Number of participants per unit area in transverse plane:

$$\begin{aligned} n_{\text{WN}}(x, y; b) &= T_A(x + b/2, y) \left[1 - \left(1 - \frac{\sigma}{B} T_B(x - b/2, y) \right)^B \right] \\ &+ T_B(x - b/2, y) \left[1 - \left(1 - \frac{\sigma}{A} T_A(x + b/2, y) \right)^A \right] \end{aligned}$$

Number of binary collisions per unit area

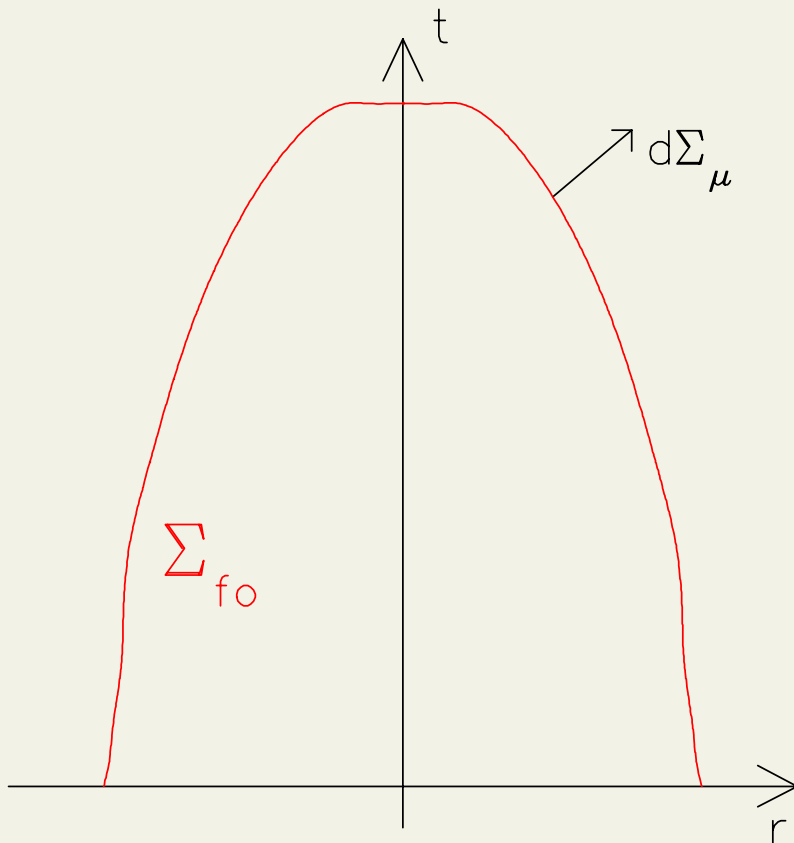
$$n_{\text{BC}}(x, y; b) = \sigma_{pp} T_A(x + b/2, y) T_B(x - b/2, y)$$

- MC-Glauber:

- sample $\rho(x, y, z)$ to get the positions of nucleons in 2 nuclei
- count # of nucleons closer than $\sqrt{\sigma_{pp}/\pi}$ in the collision
- this gives n_{WN} and n_{BC}
- repeat to get enough statistics

When to end?

- **Particles** are observed, **not fluid**
- How and when to convert fluid to particles?
- i.e. **how far is hydro valid?**



- Kinetic equilibrium requires **scattering rate** $\gg$ **expansion rate**
- **Scattering rate** $\tau_{\text{sc}}^{-1} \sim \sigma n \propto \sigma T^3$
- **Expansion rate** $\theta = \partial_\mu u^\mu$
- Fluid description breaks down when $\tau_{\text{sc}}^{-1} \approx \theta$
→ **momentum distributions freeze-out**
- $\tau_{\text{sc}}^{-1} \propto T^3 \rightarrow$ rapid transition to free streaming
- **Approximation:** decoupling takes place on **constant temperature** hypersurface Σ_{fo} , at $T = T_{\text{fo}}$

Hybrid models

- End hydro when rescatterings still frequent
- Convert fluid to particle ensembles
- Describe evolution of particles using hadronic transport
- **Advantages:**
 - chemical evolution and dissipation described
 - physical decoupling
- **Disadvantages:**
 - all the unknowns of hadronic cascade. . .
 - where and how to switch?
- **Note:** The switch from fluid to cascade is **NOT** freeze-out
⇒ particlization

Cooper-Frye

- Number of **particles emitted** = Number of **particles crossing** Σ_{fo}

$$\Rightarrow N = \int_{\Sigma_{\text{fo}}} d\Sigma_{\mu} N^{\mu}$$

- Frozen-out particles do not interact anymore: **kinetic theory**

$$\Rightarrow N^{\mu} = \int \frac{d^3\mathbf{p}}{E} p^{\mu} f(x, p \cdot u)$$

$$\Rightarrow N = \int \frac{d^3\mathbf{p}}{E} \int_{\Sigma_{\text{fo}}} d\Sigma_{\mu} p^{\mu} f(x, p \cdot u)$$

- **Invariant single inclusive momentum spectrum: (Cooper-Frye formula)**

$$E \frac{dN}{d\mathbf{p}^3} = \int_{\Sigma_{\text{fo}}} d\Sigma_{\mu} p^{\mu} f(x, p \cdot u)$$

Cooper and Frye, PRD 10, 186 (1974)

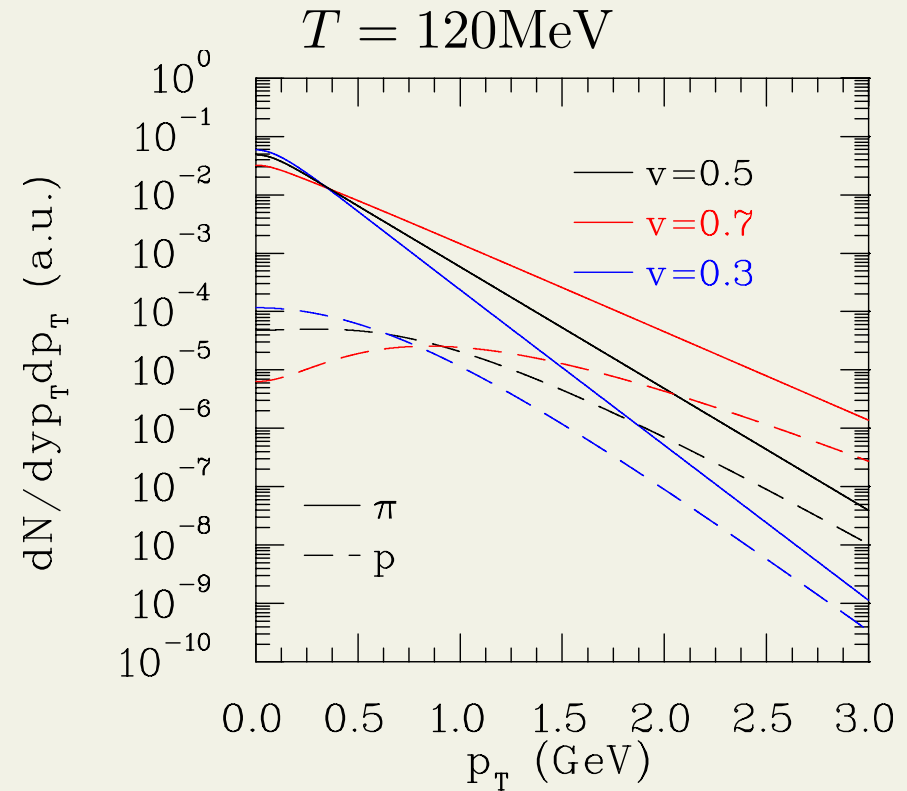
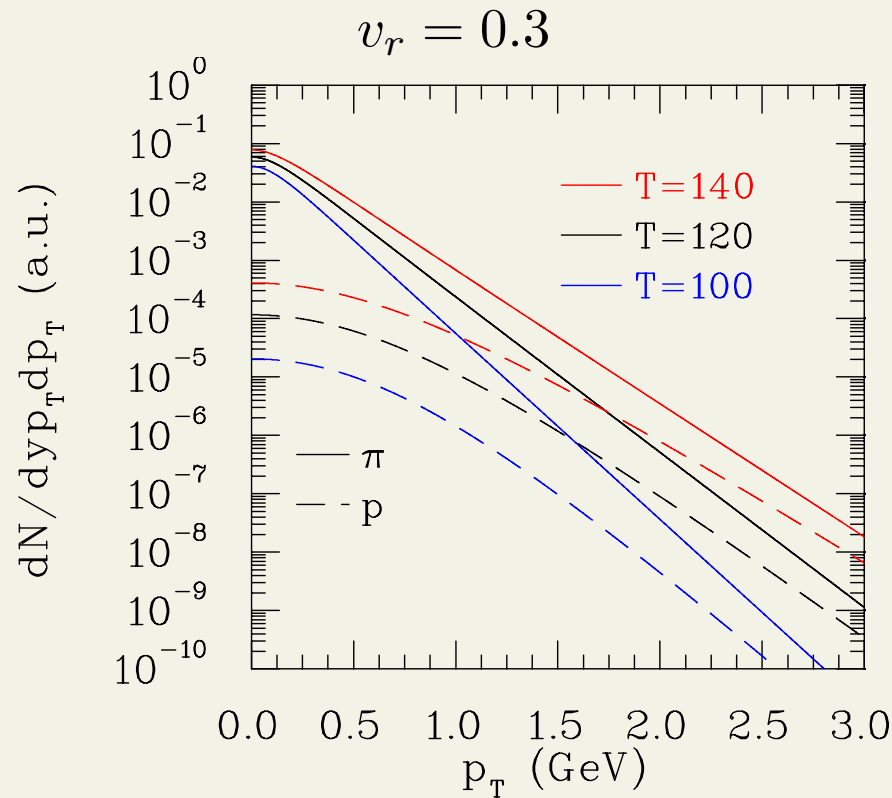
Blast wave

(Siemens and Rasmussen, PRL 42, 880 (1979))

- Freeze-out surface a **thin cylindrical shell** **radius** r , **thickness** dr , **expansion velocity** v_r , **decoupling time** τ_{fo} , **boost invariant**
- Cooper-Frye for Boltzmannions

$$\frac{dN}{dy p_T dp_T} = \frac{g}{\pi} \tau_{fo} r m_T I_0\left(\frac{v_r \gamma_r p_T}{T}\right) K_1\left(\frac{\gamma_r m_T}{T}\right)$$

effect of temperature and flow velocity



- The larger the temperature, the flatter the spectra
- The larger the velocity, the flatter the spectra $\Rightarrow$ blueshift
- The heavier the particle, the more sensitive it is to flow (shape and slope)

- Define **speed of sound** c_s :

$$c_s^2 = \left. \frac{\partial P}{\partial \epsilon} \right|_{s/n_b}$$

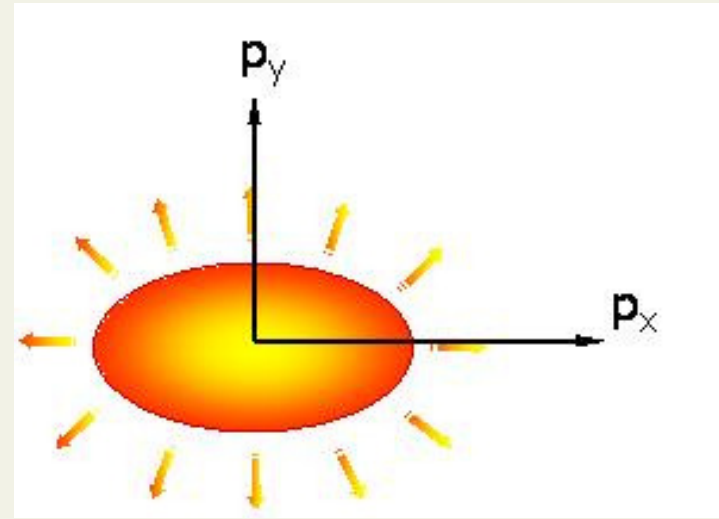
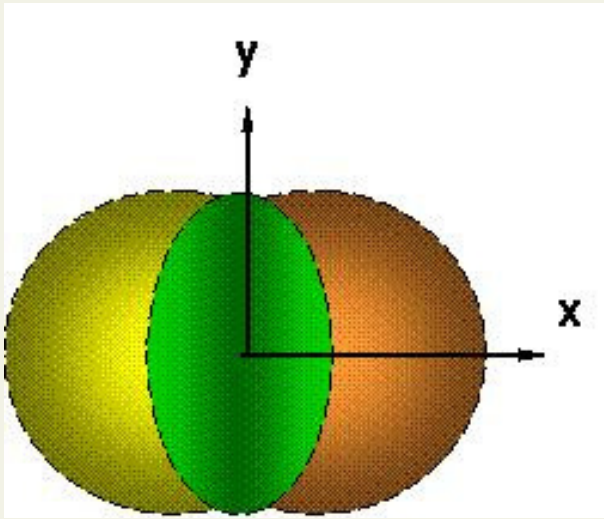
- large $c_s \Rightarrow$ **“stiff EoS”**
- small $c_s \Rightarrow$ **“soft EoS”**
- For baryon-free matter in rest frame

$$(\epsilon + P)Du^\mu = \nabla^\mu P \quad \Longleftrightarrow \quad \frac{\partial}{\partial \tau} u_\mu = -\frac{c_s^2}{s} \partial_\mu s$$

$\Rightarrow$ **The stiffer the EoS, the larger the acceleration**

Elliptic flow v_2

spatial anisotropy $\rightarrow$ final azimuthal momentum anisotropy



- Anisotropy in coordinate space + rescattering
 $\Rightarrow$ Anisotropy in momentum space

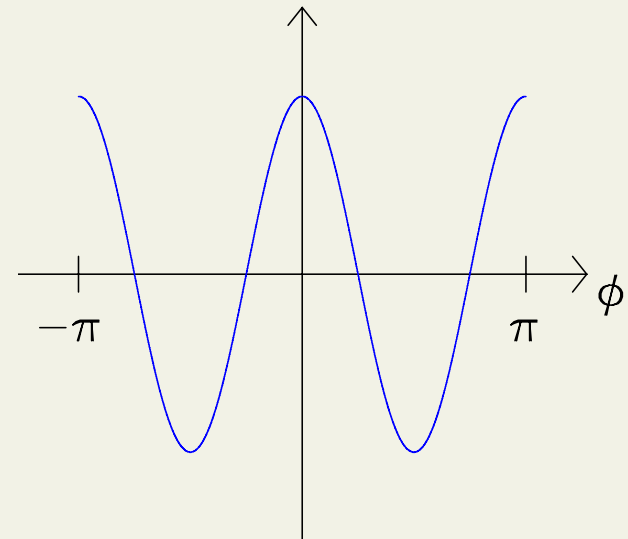
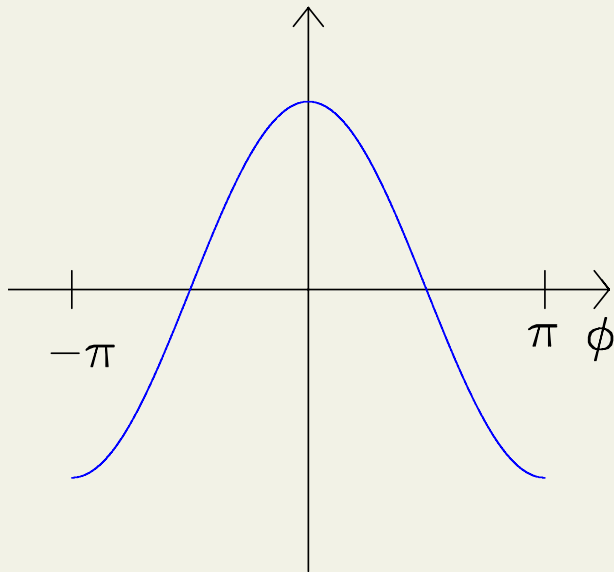
$$\frac{\partial}{\partial \tau} u_x = -\frac{c_s^2}{s} \frac{\partial}{\partial x} s \quad \text{and} \quad \frac{\partial}{\partial \tau} u_y = -\frac{c_s^2}{s} \frac{\partial}{\partial y} s$$

Elliptic flow v_2

- Fourier expansion of momentum distribution:

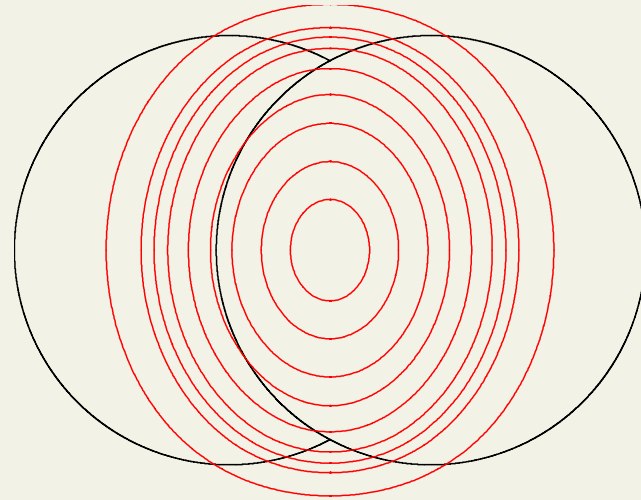
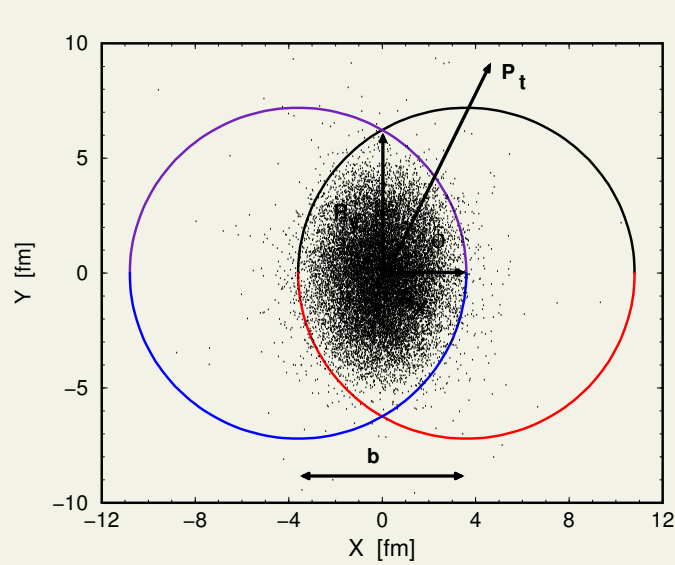
$$\frac{dN}{dy p_T dp_T d\phi} = \frac{1}{2\pi} \frac{dN}{dy p_T dp_T} (1 + 2v_1(y, p_T) \cos \phi + 2v_2(y, p_T) \cos 2\phi + \dots)$$

v_1 : **Directed flow**: preferred direction v_2 : **Elliptic flow**: preferred plane



sensitive to **speed of sound** $c_s^2 = \partial p / \partial e$ and **shear viscosity** η

Measures of anisotropy



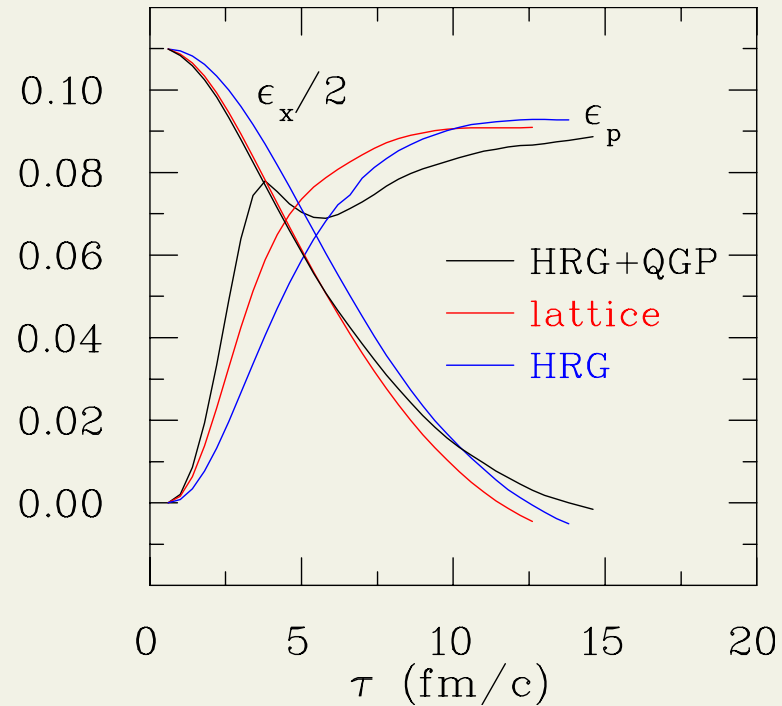
- **Spatial eccentricity**

$$\epsilon_x = \frac{\langle\langle y^2 - x^2 \rangle\rangle}{\langle\langle y^2 + x^2 \rangle\rangle} = \frac{\int dx dy \epsilon \cdot (y^2 - x^2)}{\int dx dy \epsilon \cdot (y^2 + x^2)}$$

- **Momentum anisotropy**

$$\epsilon_p = \frac{\langle T^{xx} - T^{yy} \rangle}{\langle T^{xx} + T^{yy} \rangle} = \frac{\int dx dy T^{xx} - T^{yy}}{\int dx dy T^{xx} + T^{yy}}$$

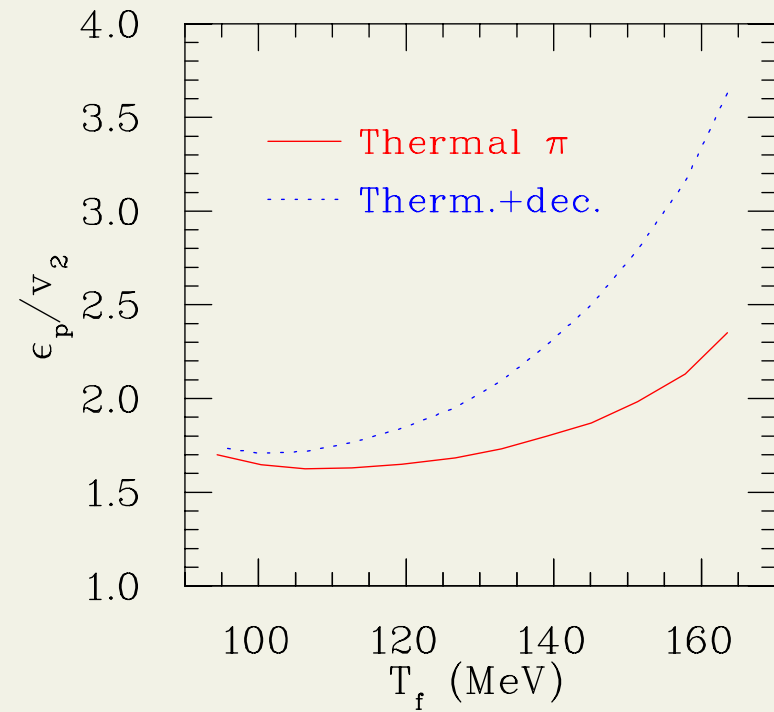
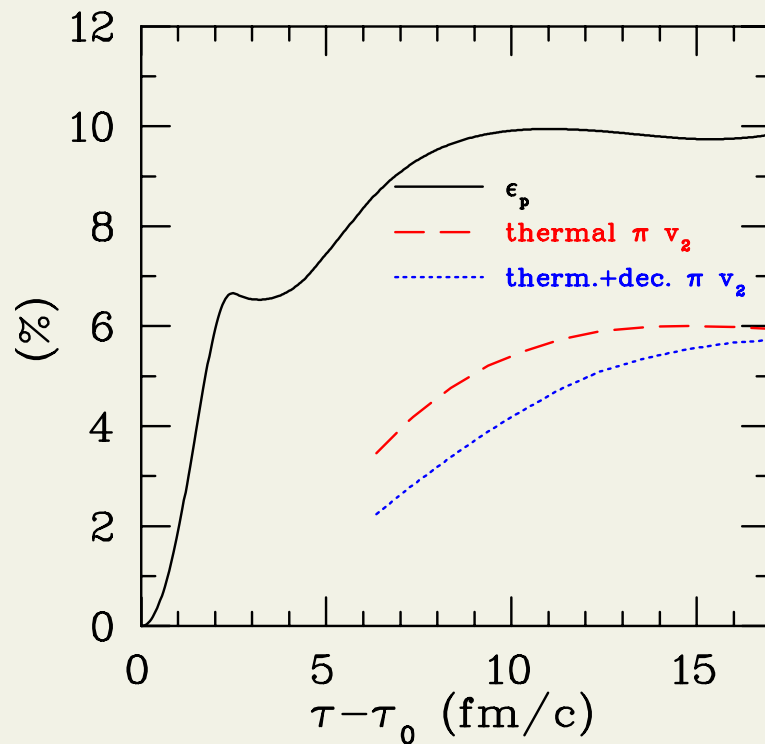
- **Au+Au @ RHIC, $b = 6$ fm:**



- ϵ_x decreases during the evolution $\Rightarrow$ **elliptic flow is self-quenching**
- Most of ϵ_p is **built up early** in the evolution

ϵ_p VS. v_2

- v_2 : not only **collective** but also **thermal** motion
- Au+Au @ RHIC, $b = 7$ fm:



- **NO** clear correspondence
- especially if one includes resonance decays

v_2

- Elliptic flow v_2 a.k.a. p_T -averaged v_2 :

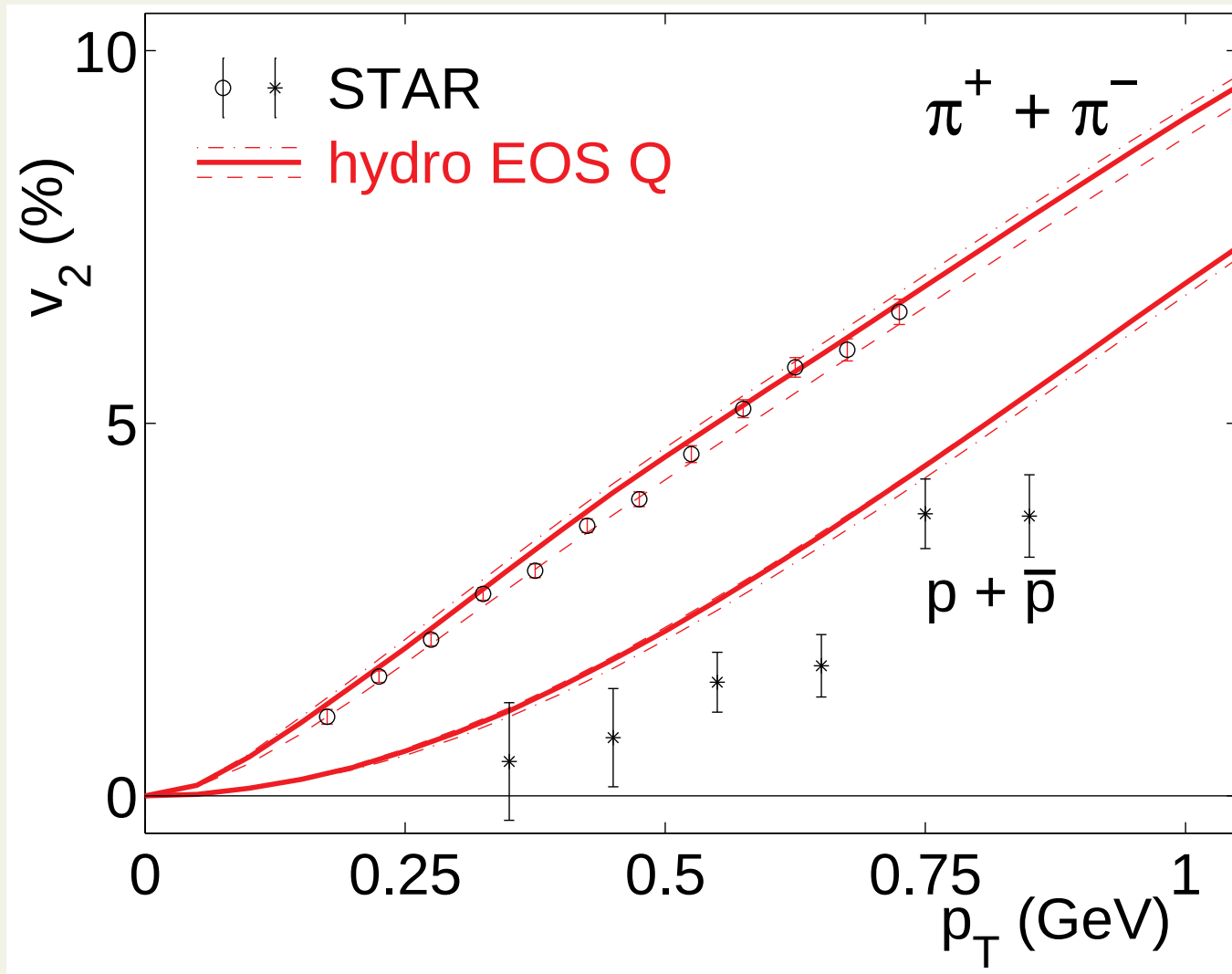
$$v_2 = \frac{\langle p_x^2 - p_y^2 \rangle}{\langle p_x^2 + p_y^2 \rangle} = \frac{\int d\phi \cos(2\phi) \frac{dN}{dy d\phi}}{dN/dy}$$

- p_T -differential v_2

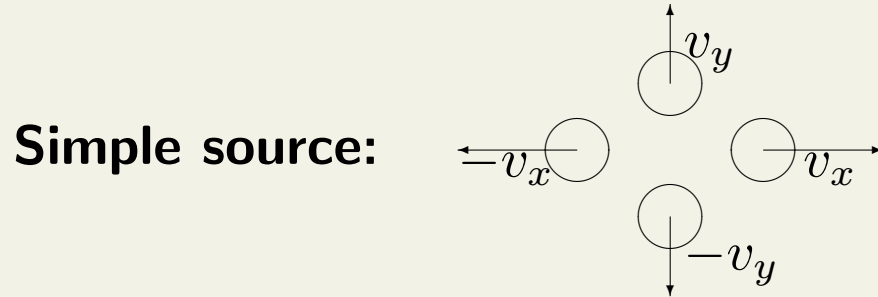
$$v_2(p_T) = \frac{\int d\phi \cos(2\phi) \frac{dN}{dy p_T dp_T d\phi}}{\int d\phi \frac{dN}{dy p_T dp_T d\phi}}$$

- If $m_1 > m_2$, $v_2(m_1) > v_2(m_2)$, but $v_2(p_T, m_1) < v_2(p_T, m_2)$!
- No contradiction, since

$$v_2 = \frac{\int dp_T v_2(p_T) \frac{dN}{dp_T}}{\int dp_T \frac{dN}{dp_T}}$$



Why $m_1 < m_2 \Rightarrow v_2(p_T, m_1) > v_2(p_T, m_2)$?

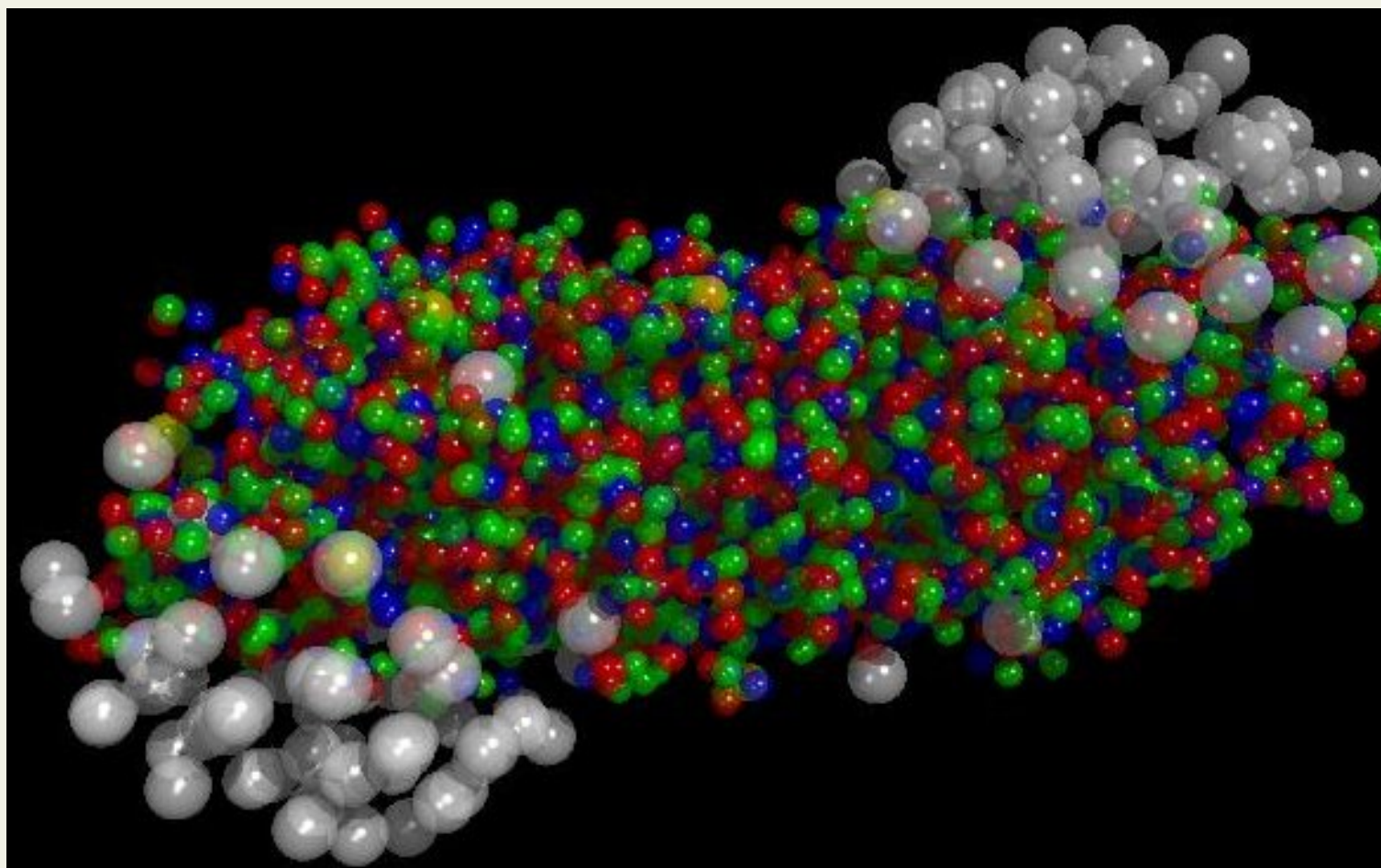


Each element has unit volume, decouples at the same time at the same temperature. Flow velocity in plane is larger than out of plane, $|v_x| > |v_y|$.

Boltzmann distribution and Cooper-Frye formula:

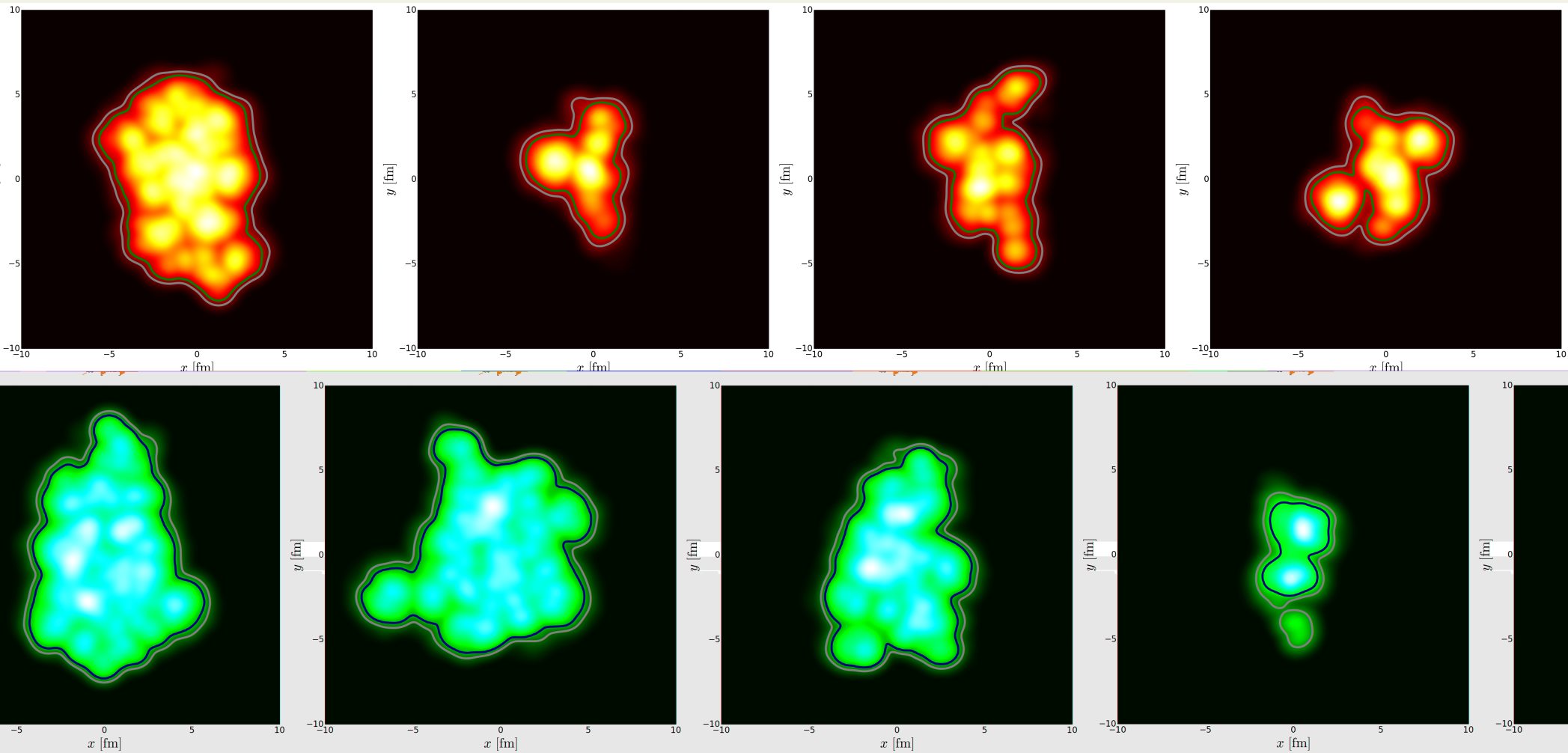
$$\begin{aligned}
 v_2(p_T) &= \frac{I_2\left(\frac{\gamma_x v_x p}{T}\right) - e^{\frac{E}{T}(\gamma_x - \gamma_y)} I_2\left(\frac{\gamma_y v_y p}{T}\right)}{I_0\left(\frac{\gamma_x v_x p}{T}\right) + e^{\frac{E}{T}(\gamma_x - \gamma_y)} I_0\left(\frac{\gamma_y v_y p}{T}\right)} \\
 &= \frac{C_1 - e^{\lambda \sqrt{m^2 + p^2}} C_2}{C_3 + e^{\lambda \sqrt{m^2 + p^2}} C_4}
 \end{aligned}$$

mass increases, numerator decreases and denominator increases
 $\rightarrow v_2$ **decreases**



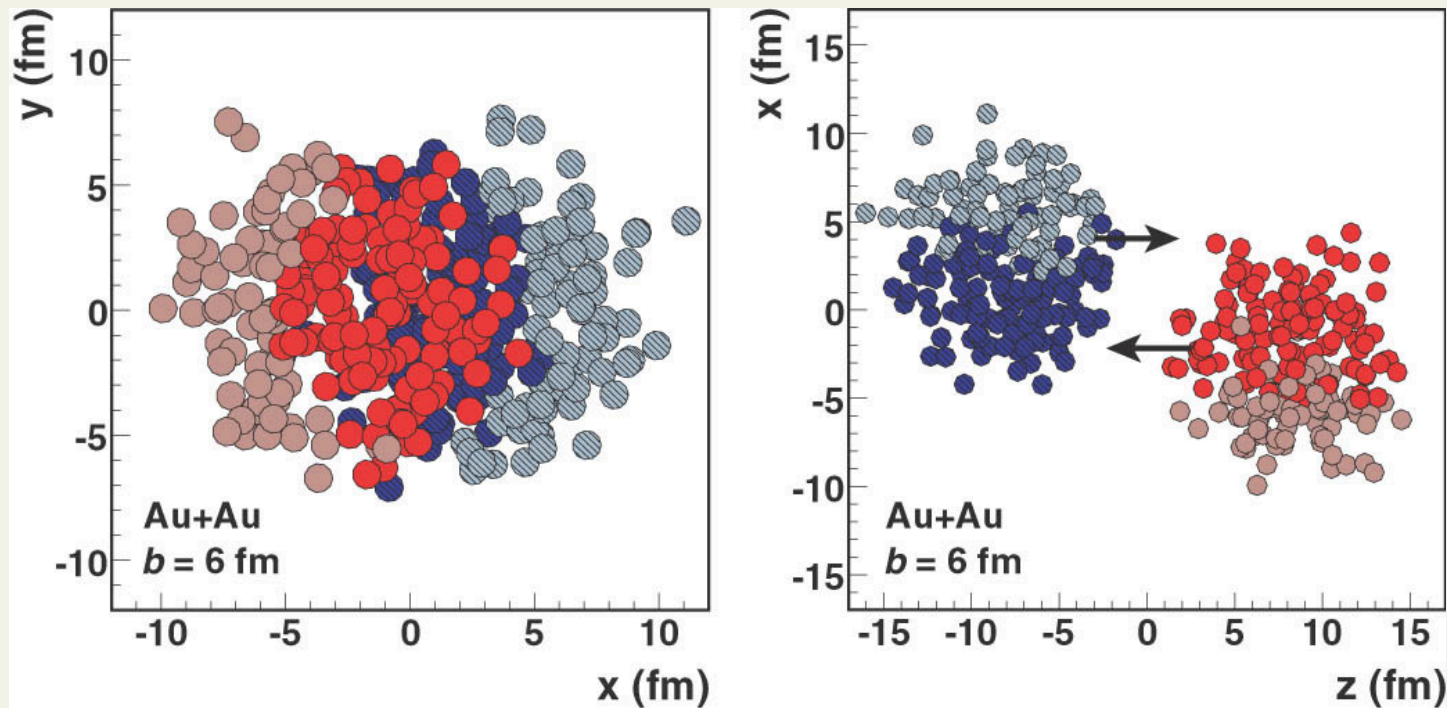
Initial state fluctuates

temperature profiles in transverse plane



©Harri Niemi

event-by-event



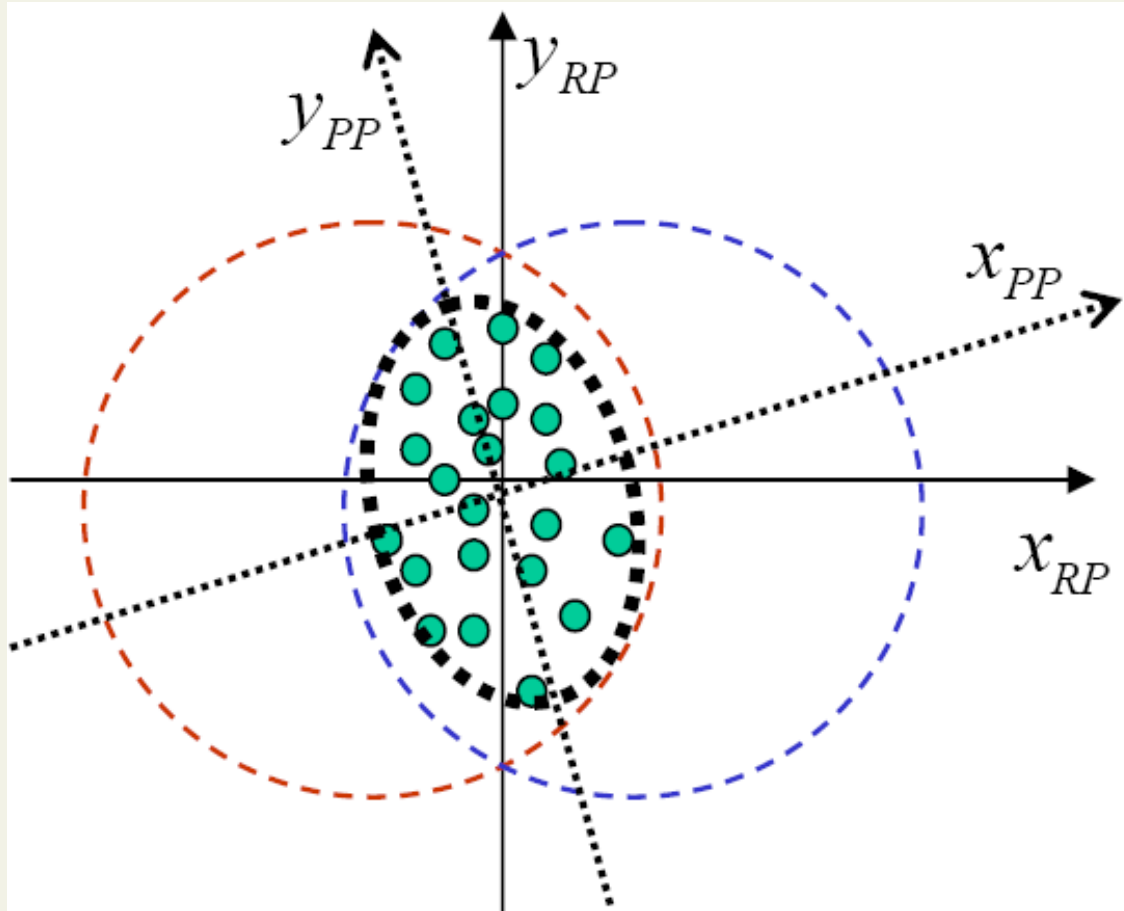
Miller *et al.*, Ann.Rev.Nucl.Part.Sci. 57, 205 (2007)

- shape fluctuates event-by-event
- all coefficients v_n finite

$$\frac{dN}{dyd\phi} = \frac{dN}{dy} \left[1 + \sum_n 2v_n \cos(2(\phi - \Psi_n)) \right]$$

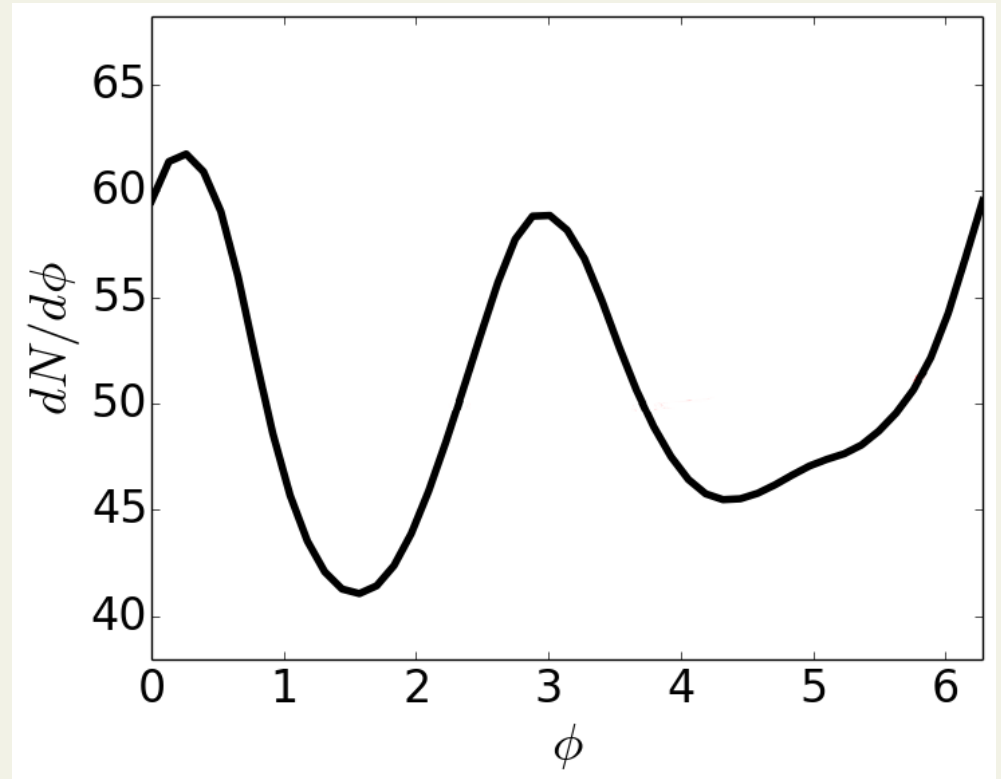
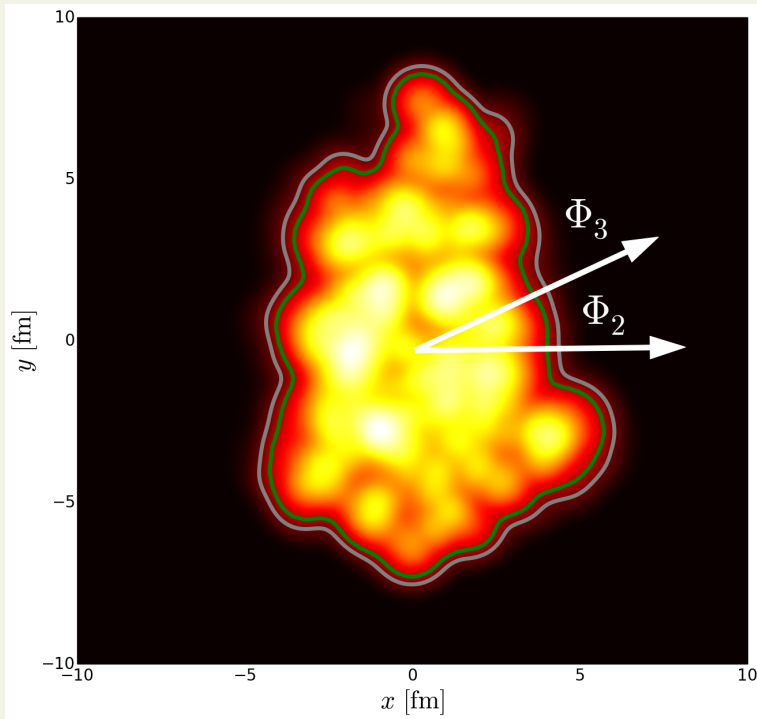
All the planes. . .

Voloshin et al. Phys. Lett. B 659, 537 (2008)



- X_{RP} : Reaction plane, spanned by beam and impact parameter
- X_{PP} : Participant plane, maximises spatial anisotropy ϵ_n
- Ψ_n : Event plane, maximises anisotropy v_n

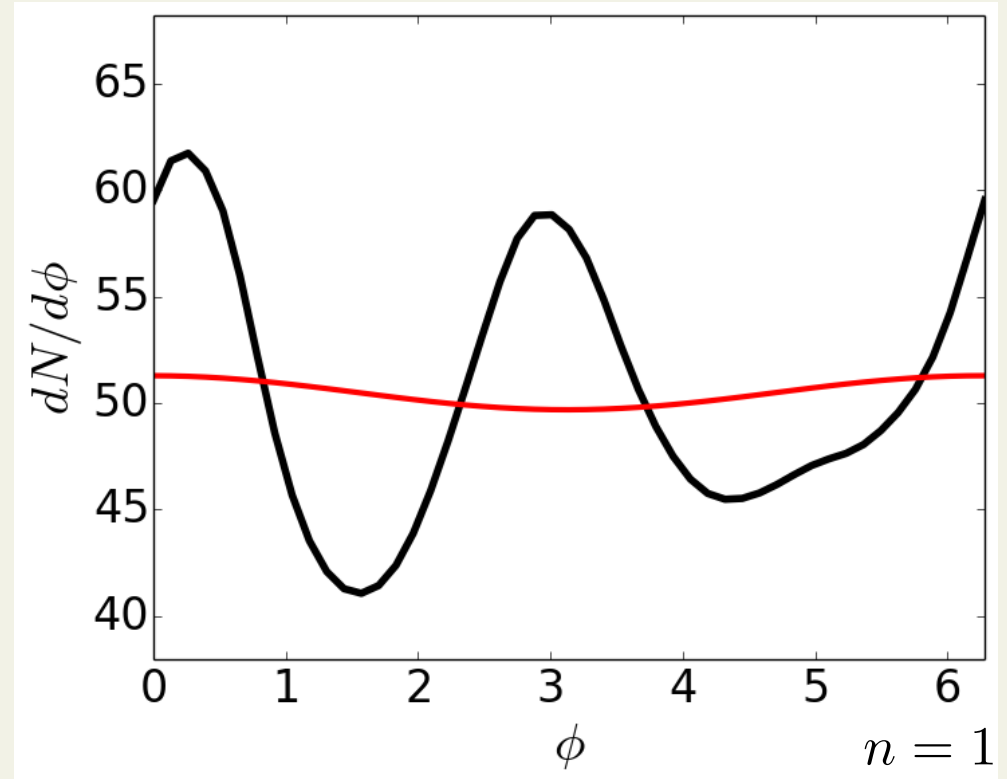
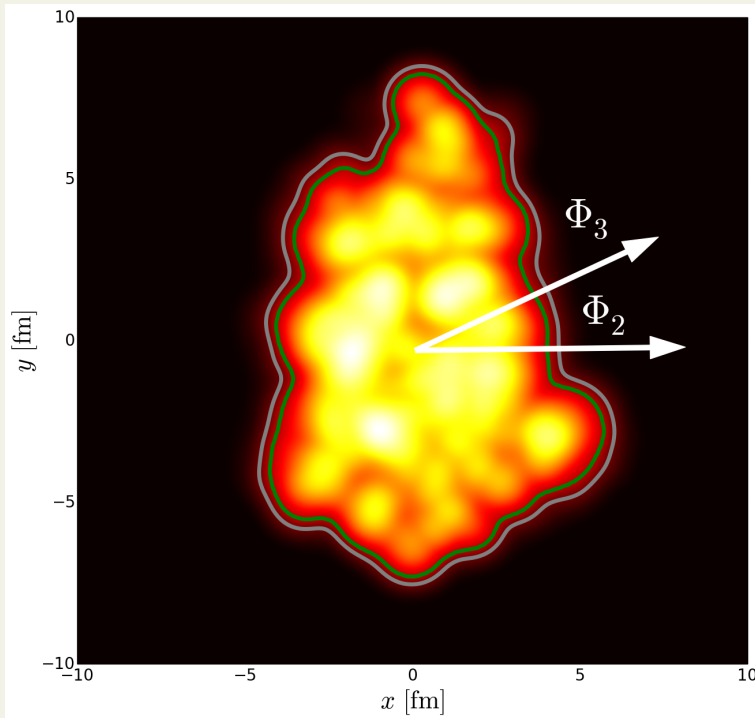
From fluid to distribution



$$\epsilon_n, \Phi_n \implies v_n, \Psi_n$$

$$\frac{dN}{dyd\phi} = \frac{dN}{dy} \left[1 + \sum_n 2v_n \cos(n(\phi - \Psi_n)) \right]$$

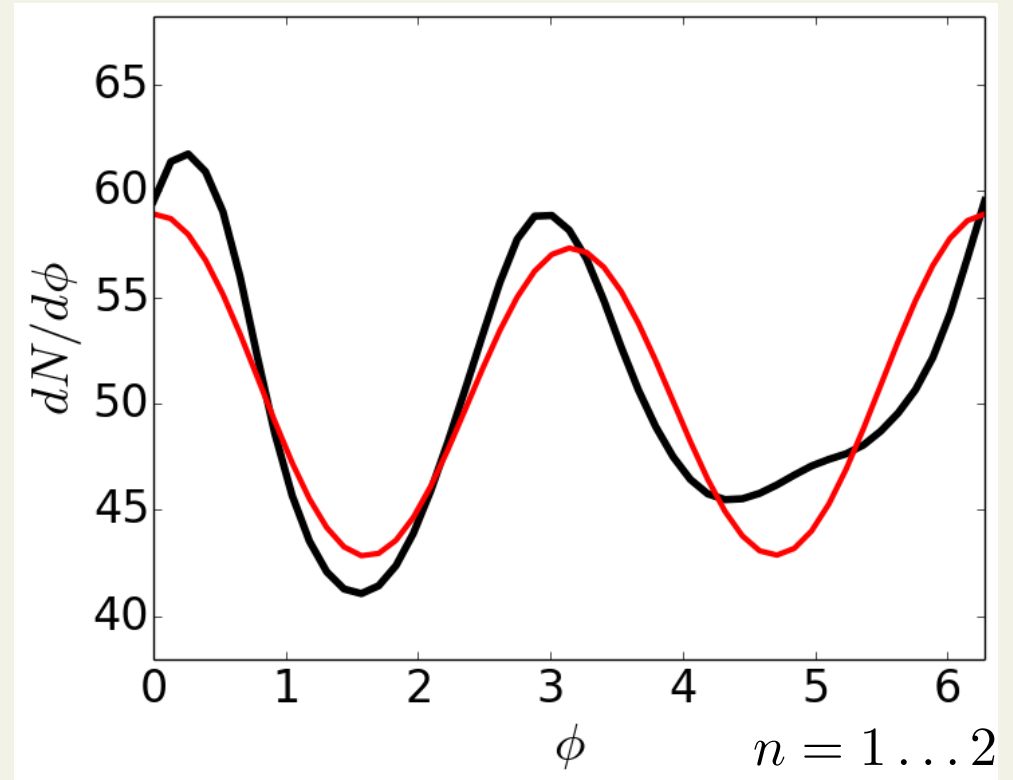
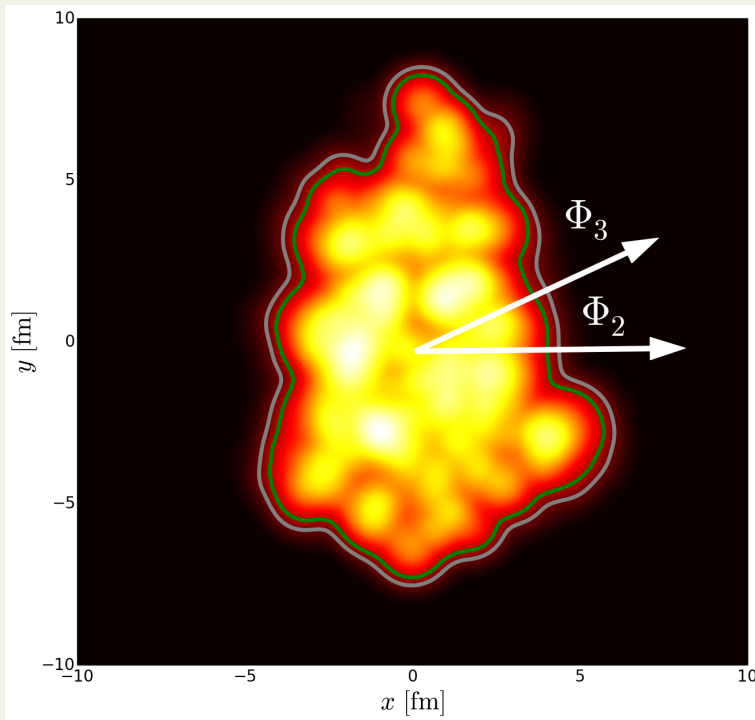
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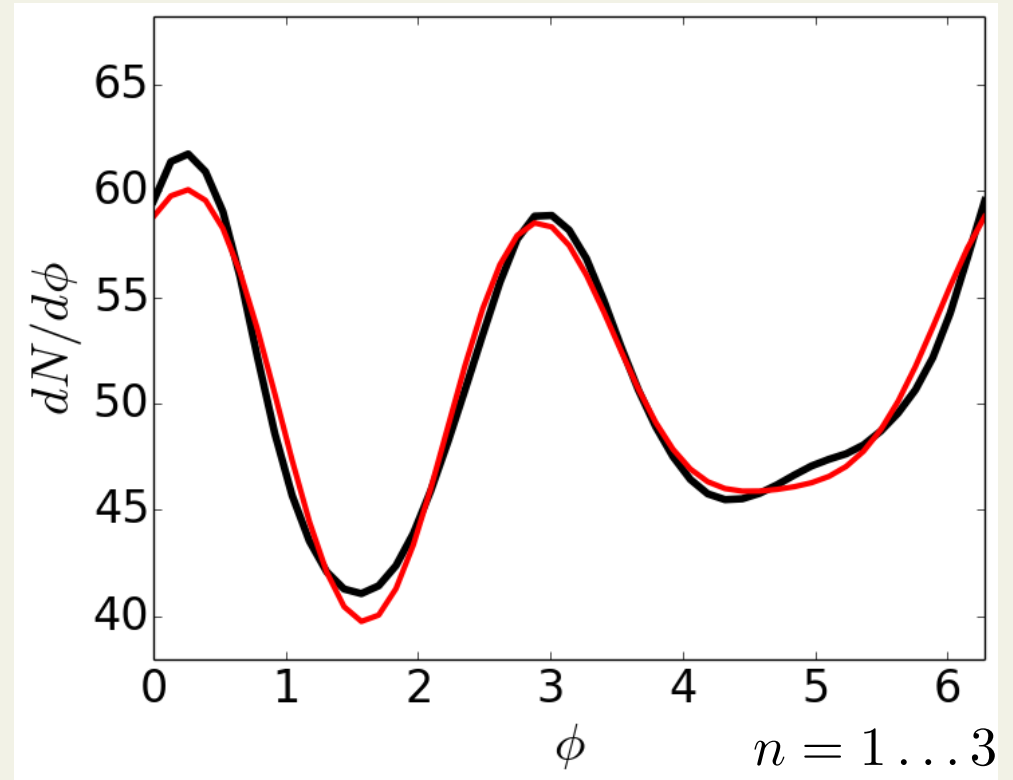
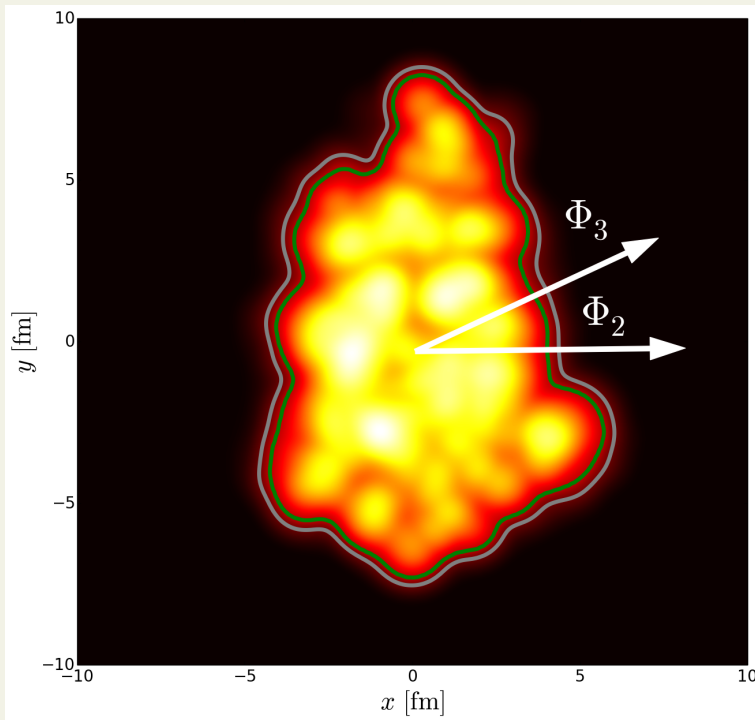
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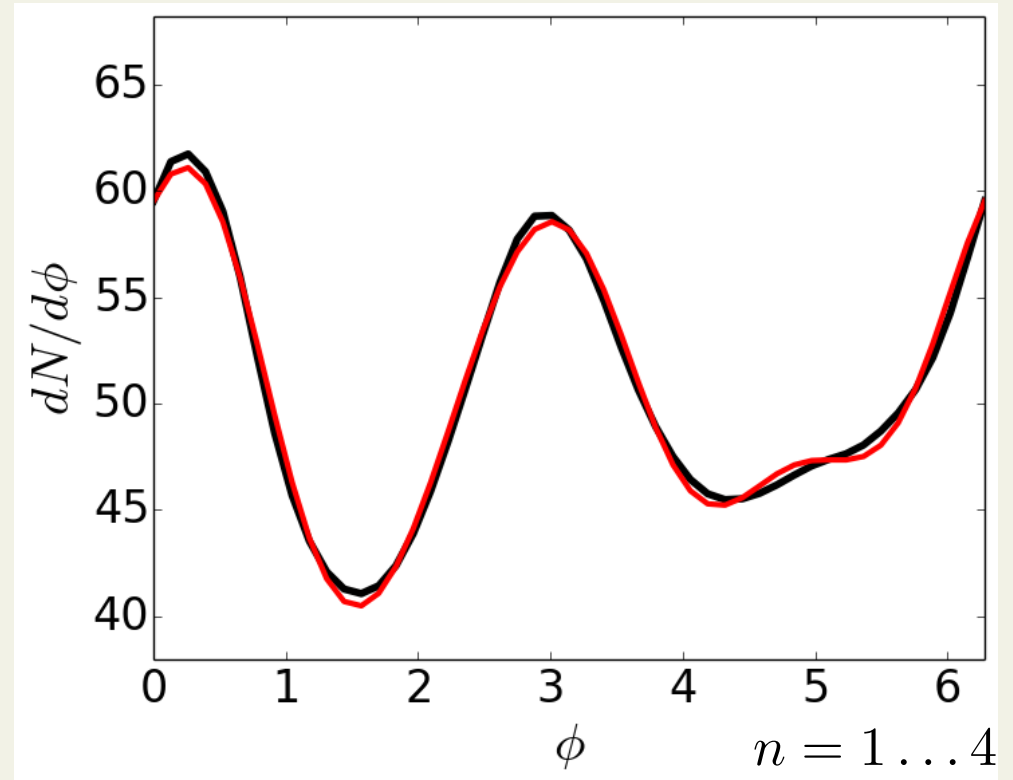
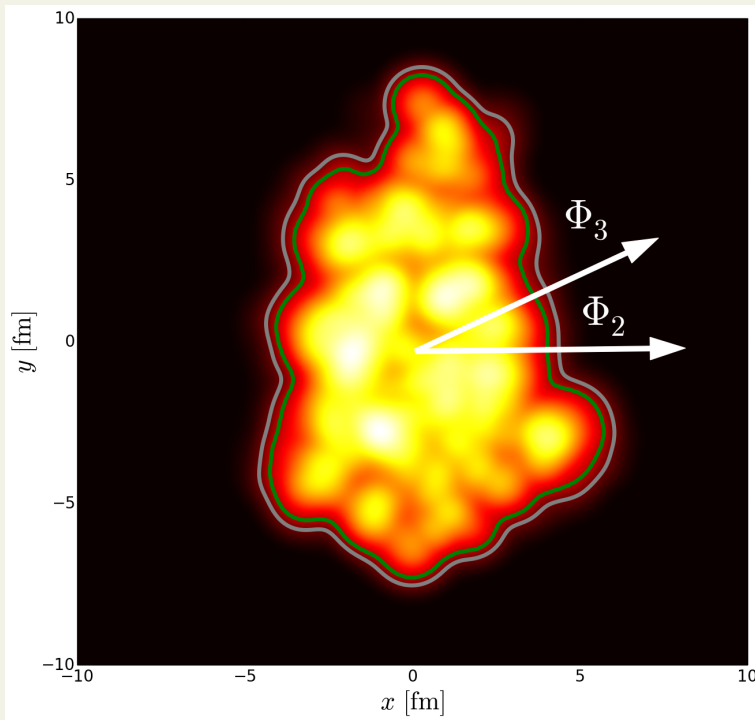
From fluid to distribution



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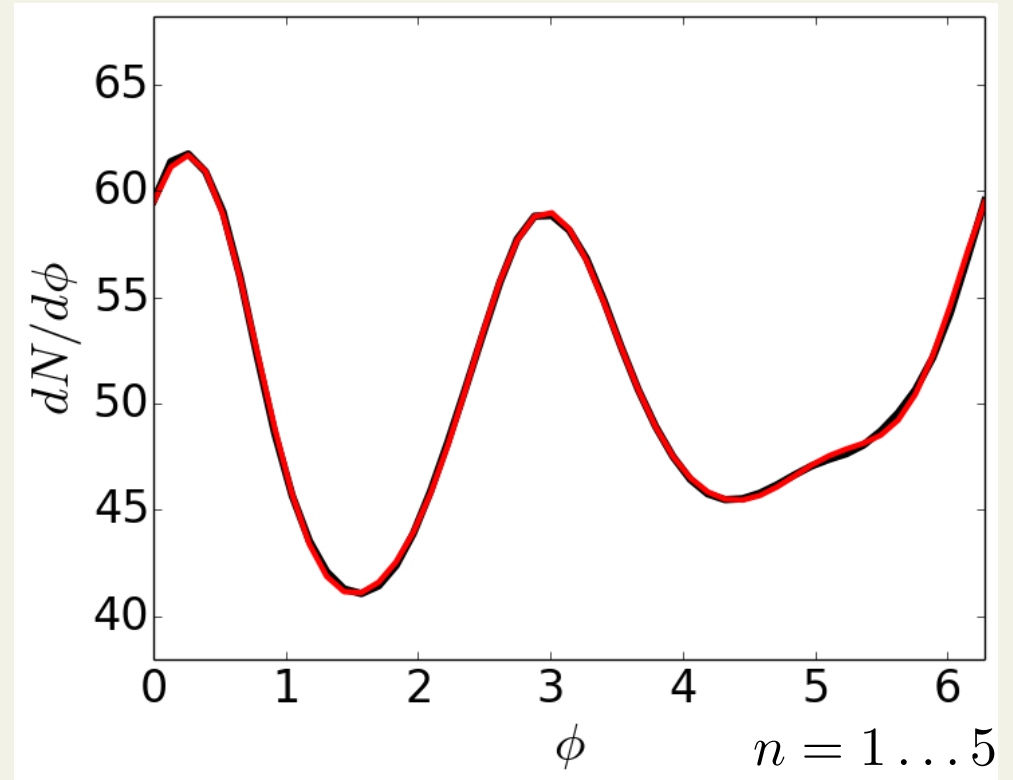
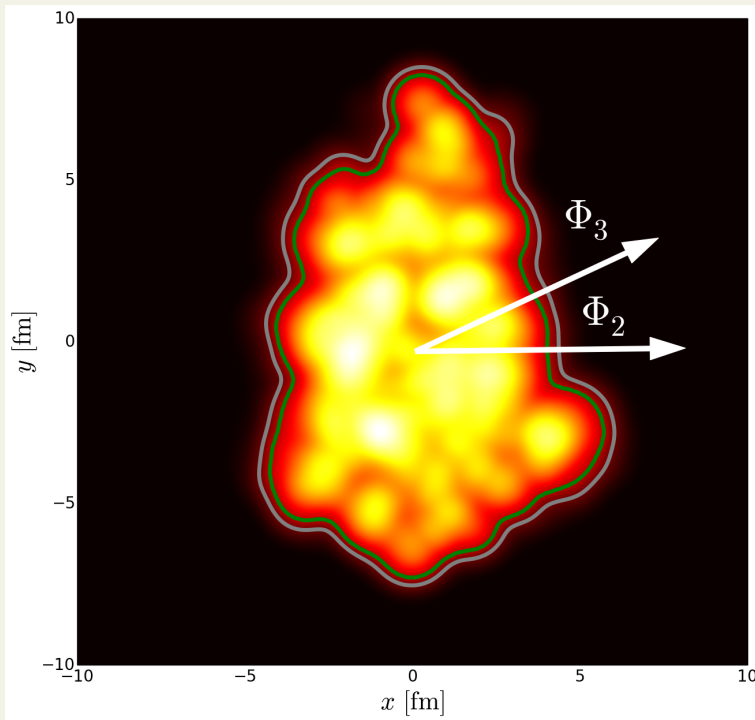
From fluid to distribution



$$\epsilon_n, \Phi_n \Rightarrow v_n, \Psi_n$$

$$\frac{dN}{dyd\phi} = \frac{dN}{dy} \left[1 + \sum_n 2v_n \cos(n(\phi - \Psi_n)) \right]$$

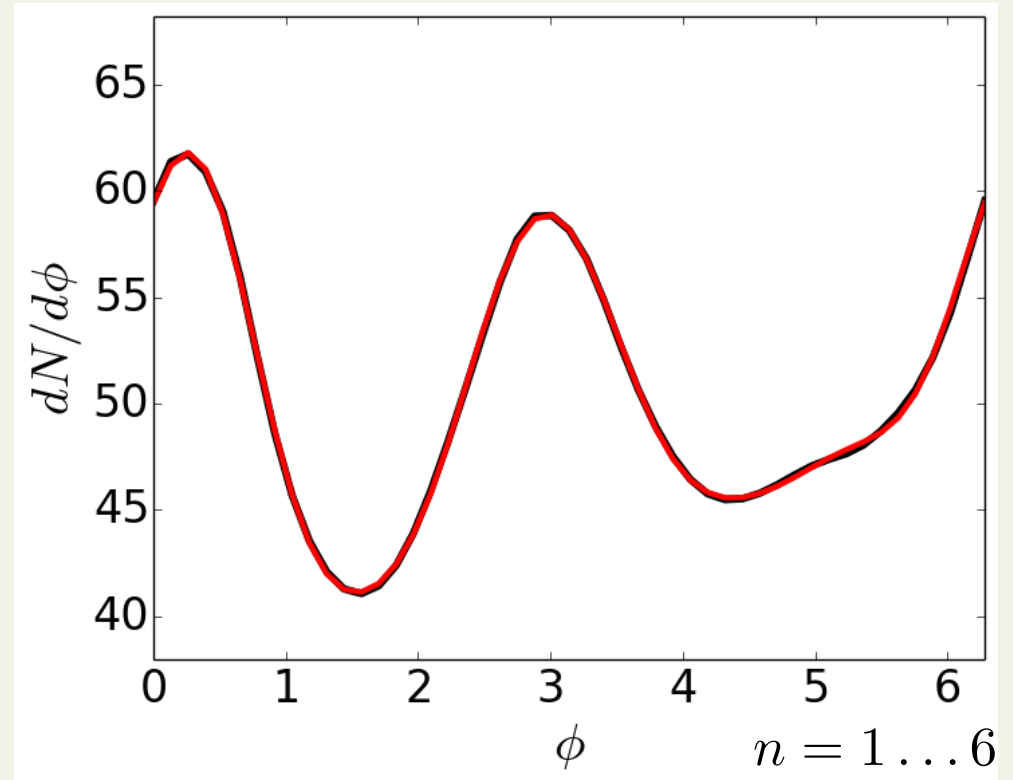
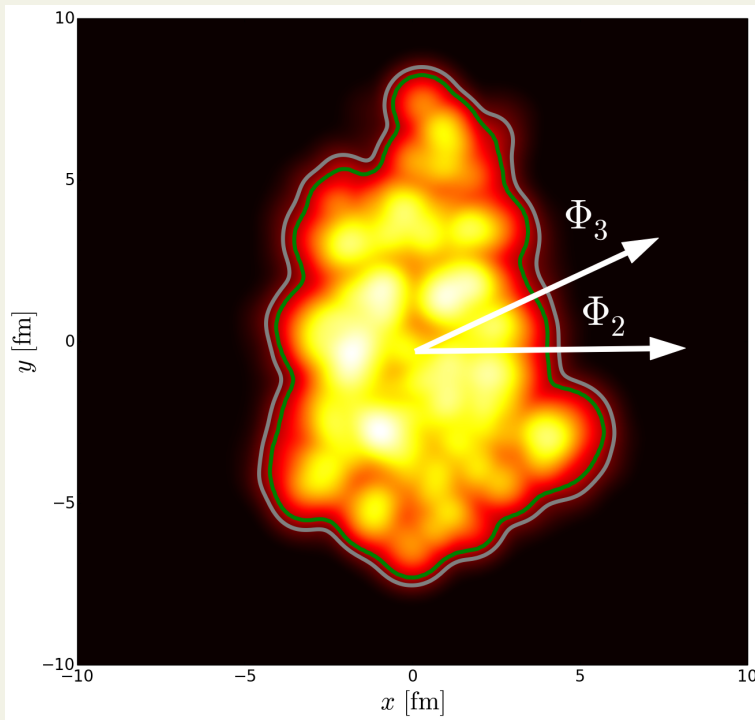
From fluid to distribution



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From fluid to distribution

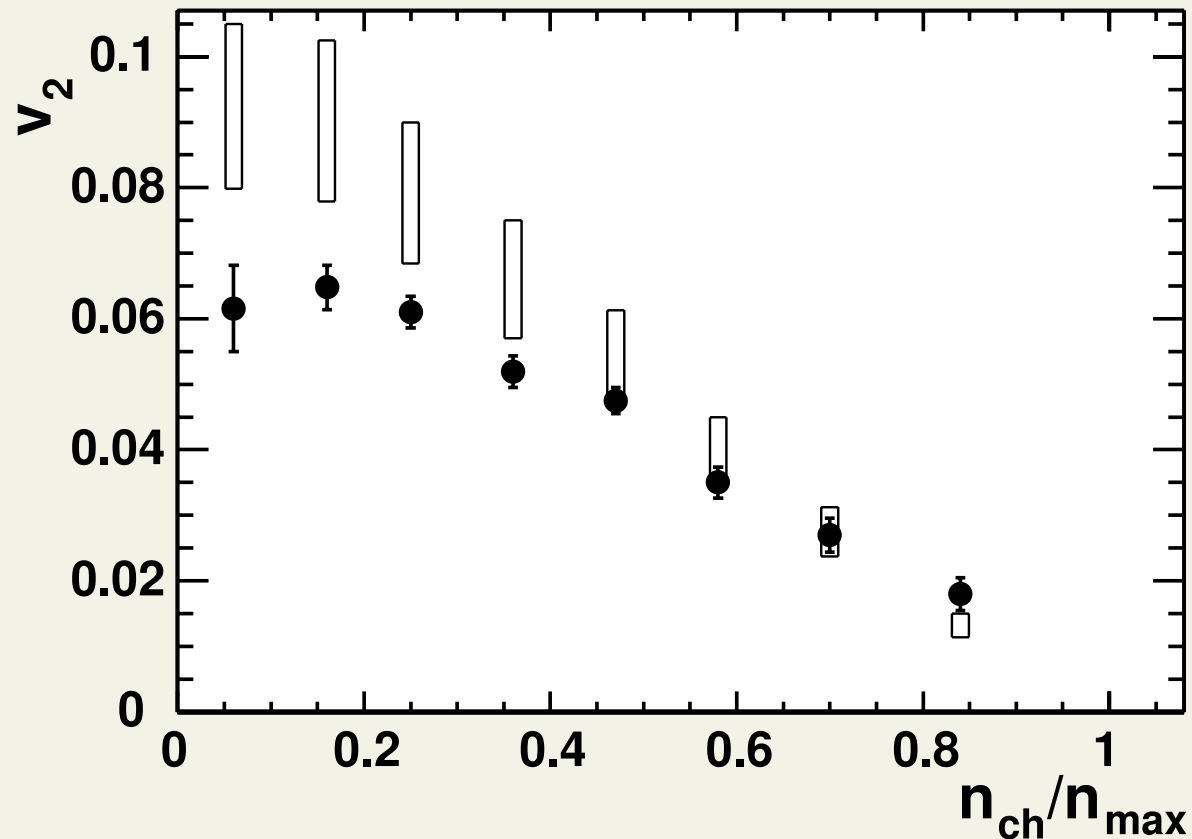


$$\epsilon_n, \Phi_n \Rightarrow v_n, \Psi_n$$

$$\frac{dN}{dyd\phi} = \frac{dN}{dy} \left[1 + \sum_n 2v_n \cos(n(\phi - \Psi_n)) \right]$$

Success of ideal hydrodynamics

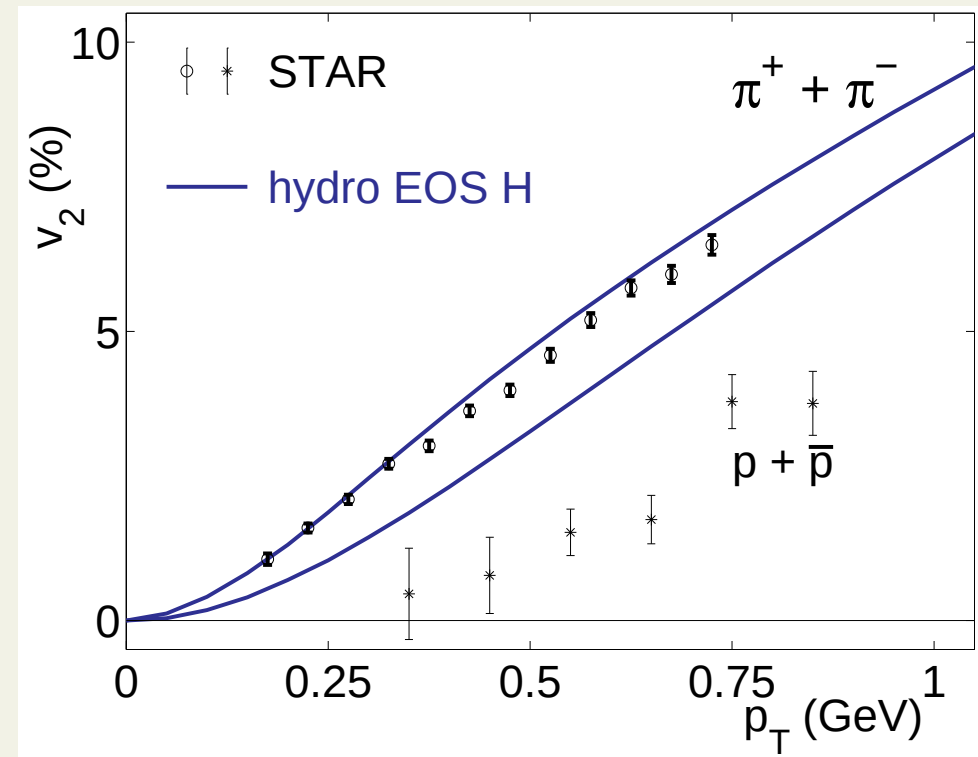
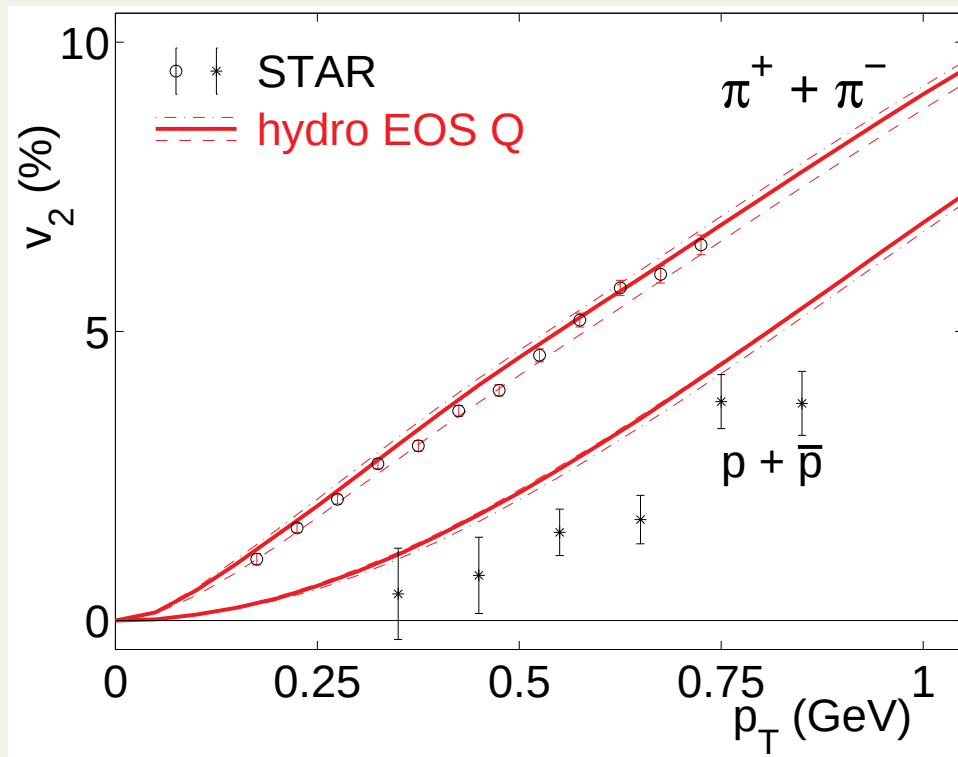
- p_T -averaged v_2 of charged hadrons:



- works beautifully in central and semi-central collisions
- but why is $v_{2,obs} > v_{2,hydro}$ in most central collisions?

Success of ideal hydrodynamics

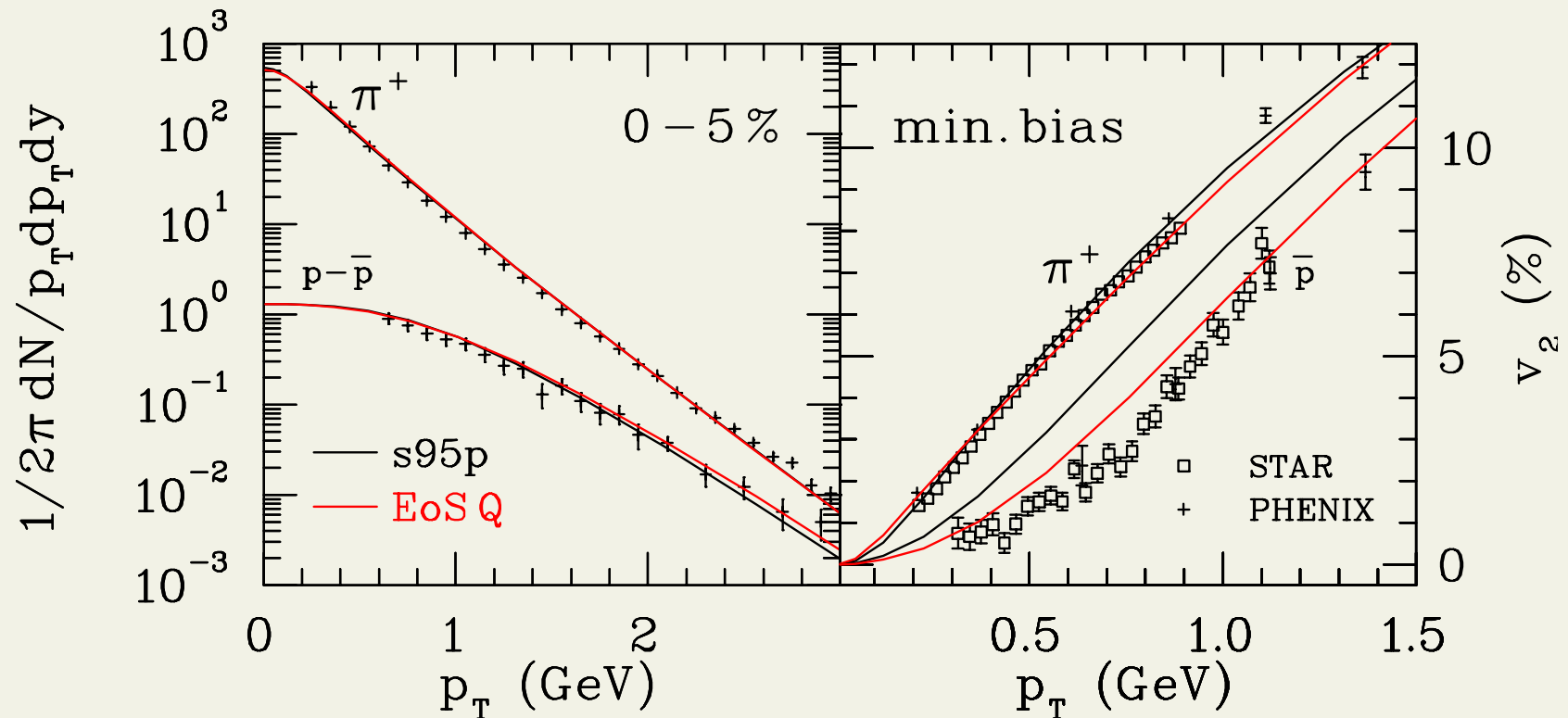
Kolb, Heinz, Huovinen et al ('01) **minbias Au+Au at RHIC**



not perfect agreement but plasma EoS favored

Lattice EoS

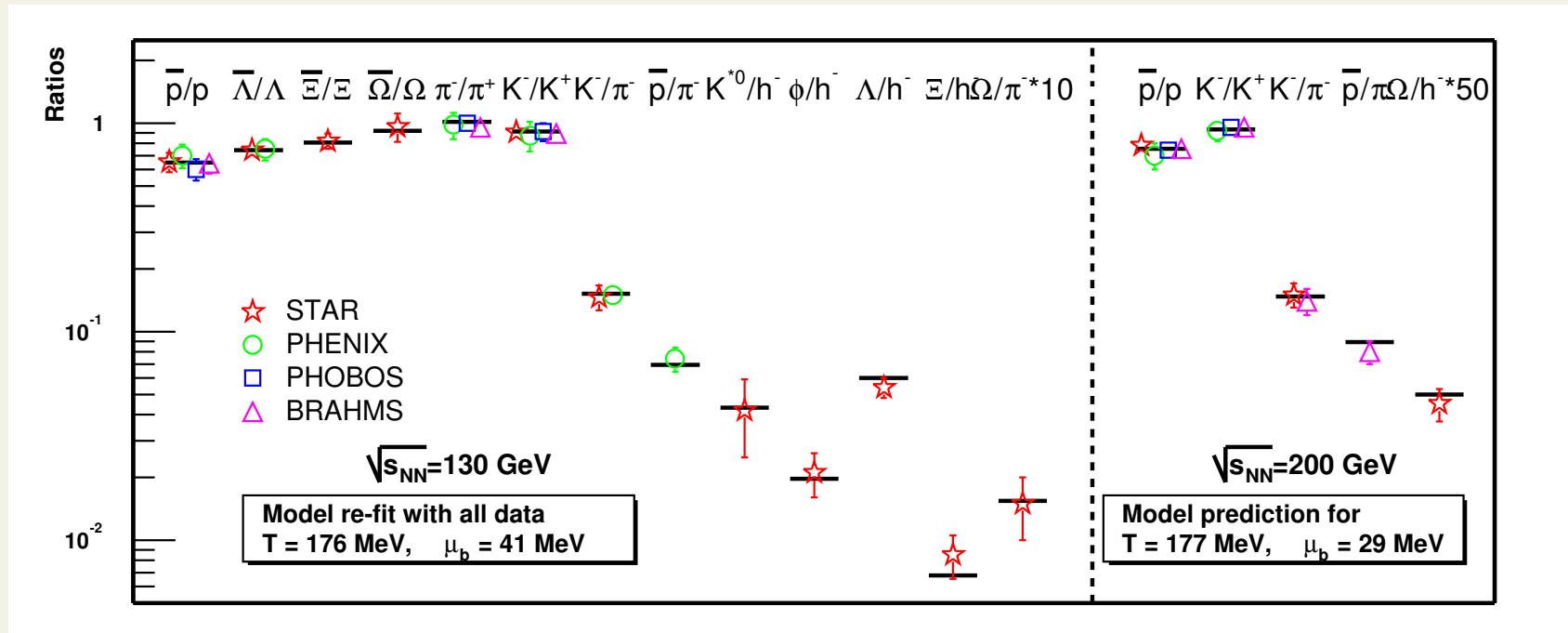
- ideal hydro, Au+Au at $\sqrt{s_{NN}} = 200$ GeV
- chemical equilibrium



- s95p: $T_{dec} = 140$ MeV
- EoS Q: first order phase transition at $T_c = 170$ MeV, $T_{dec} = 125$ MeV

Thermal models

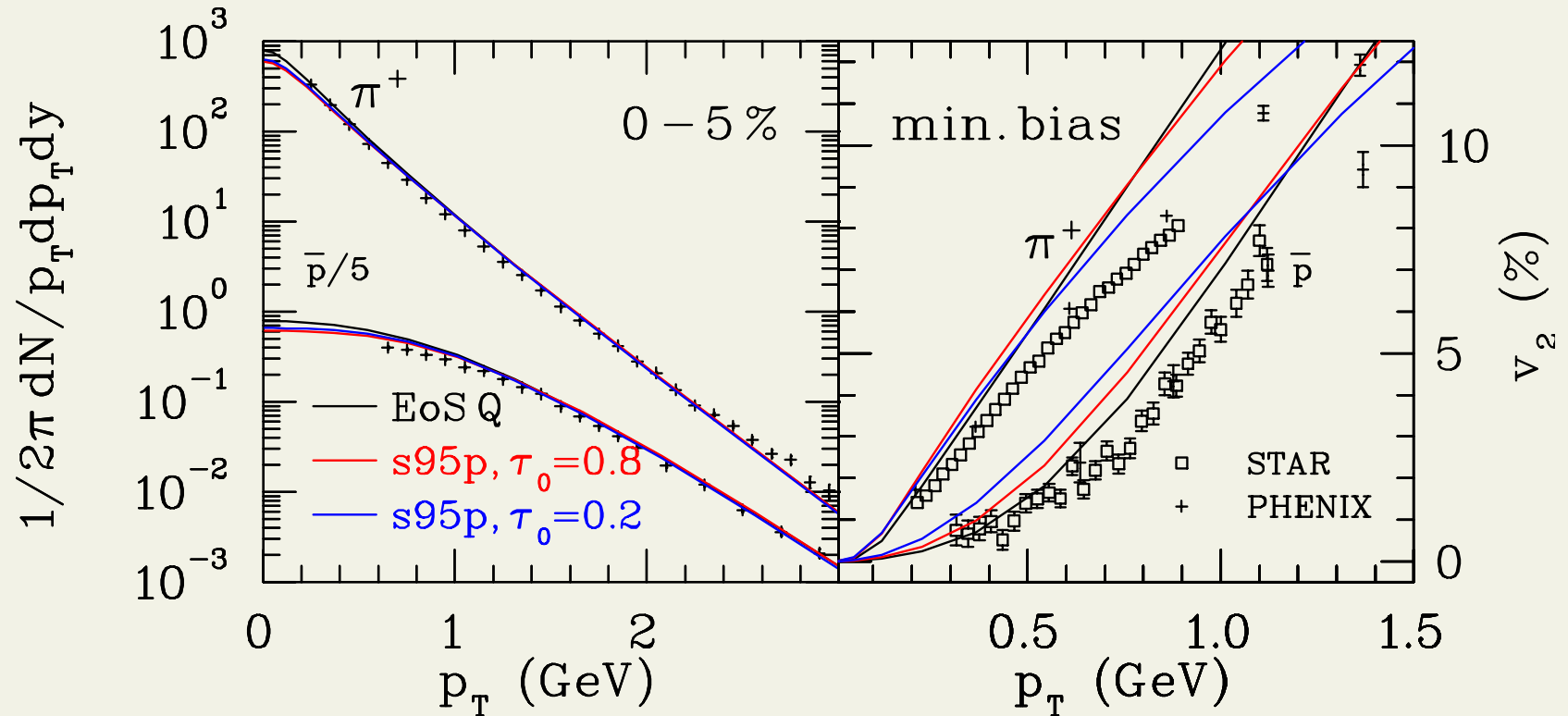
- Hadronic phase: ideal gas of massive hadrons and resonances
- in chemical equilibrium



- Particle ratios $\Longleftrightarrow T \approx 160\text{--}170$ MeV temperature
 - Evolution to $T \approx 100\text{--}140$ MeV temperature
- $\Rightarrow$ In hydro particle ratios become wrong

More realistic EoS

- ideal hydro, Au+Au at $\sqrt{s_{NN}} = 200$ GeV
- $T_{\text{chem}} = 150$ MeV



- **EoS Q:** $T_{dec} = 120$ MeV, $s_{ini} \propto N_{bin}$ $\tau_0 = 0.2$ fm/c
- **s95p, $\tau_0 = 0.8$:** $T_{dec} = 120$ MeV, $s_{ini} \propto N_{bin}$, $\tau_0 = 0.8$ fm/c
- **s95p, $\tau_0 = 0.2$:** $T_{dec} = 120$ MeV, $s_{ini} \propto N_{bin} + N_{part}$, $\tau_0 = 0.2$ fm/c

Dissipative hydrodynamics

In general

$$\begin{aligned} N^\mu &= n u^\mu + \nu^\mu \\ T^{\mu\nu} &= \epsilon u^\mu u^\nu - (P + \Pi) \Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu} \end{aligned}$$

In Landau frame,

$$W^\mu \equiv 0, \quad \nu^\mu = -\frac{q^\mu}{h} = -\frac{n}{\epsilon + P} q^\mu \quad (4)$$

and thus

$$\begin{aligned} N^\mu &= n u^\mu + \nu^\mu \\ T^{\mu\nu} &= \epsilon u^\mu u^\nu - (P_{\text{eq}} + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu} \end{aligned}$$

Dissipative hydrodynamics

In Landau frame,

$$N^\mu = nu^\mu + \nu^\mu$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P_{\text{eq}} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

Need **9 additional equations** to determine

$$\Pi, \pi^{\mu\nu}, q^\mu, P_{\text{eq}}$$

Equation of state

$$P_{\text{eq}} = P(T, \mu)$$

Matching conditions

ideal fluid $\iff$ exact local kinetic equilibrium

dissipation $\iff$ deviations from thermal distribution

Non-equilibrium thermodynamics?

- What are entropy and pressure?
- EoS? Temperature?

Matching conditions

ideal fluid $\iff$ exact local kinetic equilibrium

dissipation $\iff$ deviations from thermal distribution

Non-equilibrium thermodynamics?

- What are entropy and pressure?
- EoS? Temperature?

Energy and particle number defined for arbitrary system:

$$\epsilon = u_\mu T^{\mu\nu} u_\nu \quad \text{and} \quad n = N^\mu u_\mu$$

apply equilibrium EoS:

$$s = s_0(\epsilon, n) \quad \text{and} \quad P = P_0(\epsilon, n)$$

i.e. we match the system to an equilibrium system of *the same* ϵ and n

relativistic Navier-Stokes

Entropy four-current:

$$S^\mu = s u^\mu + \frac{\mu}{T} \frac{q^\mu}{h}$$

where

$$h = \frac{\epsilon + P}{n}$$

Require non-decrease of entropy:

$$0 \leq \partial_\mu S^\mu = -\Pi \nabla^\mu u_\mu - q_\mu \frac{T}{e + p} \nabla^\mu \frac{\mu}{T} + \pi_{\mu\nu} \nabla^{\langle\mu} u^{\nu\rangle}$$

where

$$A^{\langle\mu\nu\rangle} = \left[\frac{1}{2} (\Delta_\sigma^\mu \Delta_\tau^\nu + \Delta_\tau^\nu \Delta_\sigma^\mu) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\sigma\tau} \right] A^{\sigma\tau}$$

and

$$\nabla^\mu = \Delta^{\mu\nu} \partial_\nu$$

relativistic Navier-Stokes

$$0 \leq \partial_\mu S^\mu = \Pi X + q_\mu X^\mu + \pi_{\mu\nu} X^{\mu\nu}$$

is always valid if we identify

$$\Pi \propto X, \quad q^\mu \propto X^\mu, \quad \pi^{\mu\nu} \propto X^{\mu\nu}$$

dissipative currents small corrections linear in gradients

$$\begin{aligned} \Pi &= -\zeta \nabla^\mu u_\mu \\ q^\mu &= -\kappa \frac{T}{e + p} \nabla^\mu \frac{\mu}{T} \\ \pi^{\mu\nu} &= 2\eta \nabla^{\langle\mu} u^{\nu\rangle} \end{aligned}$$

η, ζ shear and bulk viscosities, κ heat conductivity

Navier-Stokes equations of motion

$$Dn = -n\partial_\mu u^\mu - \partial_\mu \left(\kappa \frac{Tn}{h^2} \nabla^\mu \frac{\mu}{T} \right)$$

$$D\epsilon = -(\epsilon + P - \zeta \nabla^\alpha u_\alpha) \partial_\mu u^\mu + 2\eta \nabla^{\langle \alpha} u^{\beta \rangle} \nabla_{\langle \alpha} u_{\beta \rangle}$$

$$(\epsilon + P - \zeta \nabla^\alpha u_\alpha) Du^\mu = \nabla^\mu (P - \zeta \nabla^\alpha u_\alpha) - 2\Delta_\alpha^\mu \partial_\beta (\eta \nabla^{\langle \alpha} u^{\beta \rangle})$$

where

$$D = u^\mu \partial_\mu \quad \text{and} \quad \nabla^\mu = \Delta^{\mu\nu} \partial_\nu$$

Navier-Stokes equations of motion

$$Dn = -n\partial_\mu u^\mu - \partial_\mu \left(\kappa \frac{Tn}{h^2} \nabla^\mu \frac{\mu}{T} \right)$$

$$D\epsilon = -(\epsilon + P - \zeta \nabla^\alpha u_\alpha) \partial_\mu u^\mu + 2\eta \nabla^{\langle \alpha} u^{\beta \rangle} \nabla_{\langle \alpha} u_{\beta \rangle}$$

$$(\epsilon + P - \zeta \nabla^\alpha u_\alpha) D u^\mu = \nabla^\mu (P - \zeta \nabla^\alpha u_\alpha) - 2\Delta_\alpha^\mu \partial_\beta (\eta \nabla^{\langle \alpha} u^{\beta \rangle})$$

where

$$D = u^\mu \partial_\mu \quad \text{and} \quad \nabla^\mu = \Delta^{\mu\nu} \partial_\nu$$

but these are parabolic. . .

Parabolic partial differential equations

PDE of the form

$$A \frac{\partial^2}{\partial x^2} u + B \frac{\partial^2}{\partial x \partial y} u + C \frac{\partial^2}{\partial y^2} u + D \frac{\partial}{\partial x} u + E \frac{\partial}{\partial y} u + F = 0$$

is parabolic if

$$B^2 - AC = 0$$

Such equations provide **infinite speed for signal propagation**

Müller ('76), Israel & Stewart ('79) ...

Solutions are **unstable**

Hiscock & Lindblom, PRD31, 725 (1985) ...

Hyperbolic partial differential equations

PDE of the form

$$A \frac{\partial^2}{\partial x^2} u + B \frac{\partial^2}{\partial x \partial y} u + C \frac{\partial^2}{\partial y^2} u + D \frac{\partial}{\partial x} u + E \frac{\partial}{\partial y} u + F = 0$$

is hyperbolic if

$$B^2 - AC > 0$$

For example one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Solutions stable and with finite propagation speed.

Causal viscous hydro

To obtain causal equations we have to replace

$$\Pi = -\zeta \nabla^\mu u_\mu$$

by

$$\tau_\Pi D\Pi + \Pi = -\zeta \nabla^\mu u_\mu + \dots$$

or something similar.

Causal viscous hydro

Israel & Stewart:

Entropy four-flow including terms second order in dissipative fluxes:

$$S^\mu = su^\mu + \frac{\mu}{T} \frac{q^\mu}{h} - (\beta_0 \Pi^2 - \beta_1 q_\nu q^\nu + \beta_2 \pi_{\lambda\nu} \pi^{\lambda\nu}) \frac{u^\mu}{2T} - \frac{\alpha_0 q^\mu \Pi}{T} + \frac{\alpha_1 q_\nu \pi^{\nu\mu}}{T}$$

$\Rightarrow$ “Second order theory”

or, rather, **Transient fluid dynamics**

Evolution equation for shear

Require non-decrease of entropy:

$$0 \leq \partial_\mu S^\mu = \Pi X + q_\mu X^\mu + \pi_{\mu\nu} X^{\mu\nu}$$

Identify $\pi^{\mu\nu} = 2\eta X^{\langle\mu\nu\rangle}$:

$$\begin{aligned} \pi^{\mu\nu} = & 2\eta \left[\nabla^{\langle\mu} u^{\nu\rangle} - \beta_2 \langle u^\lambda \partial_\lambda \pi^{\mu\nu} \rangle - \frac{1}{2} \pi^{\mu\nu} T \partial_\lambda \left(\frac{\tau_\pi u^\lambda}{2\eta T} \right) \right] \\ & + 2\eta \left[\alpha_1 \nabla^{\langle\mu} q^{\nu\rangle} + a'_1 q^{\langle\mu} u^\lambda \partial_\lambda u^{\nu\rangle} \right] \end{aligned}$$

where

$$A^{\langle\mu\nu\rangle} = \left[\frac{1}{2} (\Delta_\sigma^\mu \Delta_\tau^\nu + \Delta_\tau^\nu \Delta_\sigma^\mu) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\sigma\tau} \right] A^{\sigma\tau}$$

and

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu.$$

Israel-Stewart evolution equations

$$D\Pi = -\frac{1}{\tau_\Pi} (\Pi + \zeta \nabla_\mu u^\mu) - \frac{1}{2} \Pi \left(\nabla_\mu u^\mu + D \ln \frac{\beta_0}{T} \right)$$

$$+ \frac{\alpha_0}{\beta_0} \partial_\mu q^\mu - \frac{a'_0}{\beta_0} q^\mu D u_\mu$$

$$Dq^\mu = -\frac{1}{\tau_q} \left[q^\mu + \kappa_q \frac{T^2 n}{\varepsilon + p} \nabla^\mu \left(\frac{\mu}{T} \right) \right] - u^\mu q_\nu D u^\nu$$

$$- \frac{1}{2} q^\mu \left(\nabla_\lambda u^\lambda + D \ln \frac{\beta_1}{T} \right) - \omega^{\mu\lambda} q_\lambda$$

$$- \frac{\alpha_0}{\beta_1} \nabla^\mu \Pi + \frac{\alpha_1}{\beta_1} (\partial_\lambda \pi^{\lambda\mu} + u^\mu \pi^{\lambda\nu} \partial_\lambda u_\nu) + \frac{a_0}{\beta_1} \Pi D u^\mu - \frac{a_1}{\beta_1} \pi^{\lambda\mu} D u_\lambda$$

$$D\pi^{\mu\nu} = -\frac{1}{\tau_\pi} \left(\pi^{\mu\nu} - 2\eta \nabla^{\langle\mu} u^{\nu\rangle} \right) - (\pi^{\lambda\mu} u^\nu + \pi^{\lambda\nu} u^\mu) D u_\lambda$$

$$- \frac{1}{2} \pi^{\mu\nu} \left(\nabla_\lambda u^\lambda + D \ln \frac{\beta_2}{T} \right) - 2\pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda}$$

$$- \frac{\alpha_1}{\beta_2} \nabla^{\langle\mu} q^{\nu\rangle} + \frac{a'_1}{\beta_2} q^{\langle\mu} D u^{\nu\rangle}$$

Israel-Stewart evolution. . .

bulk pressure Π , shear stress $\pi^{\mu\nu}$ heat flow q^μ treated as independent dynamical quantities that **relax** to their Navier-Stokes value on time scales $\tau_\Pi(e, n)$, $\tau_\pi(e, n)$, $\tau_q(e, n)$

Equations of motion	5 equations
evolution of bulk	1 equation
evolution of heat flow	3 equations
evolution of shear stress	5 equations
14 equations, 14 unknowns	

These equations are causal and stable

But what are the **parameters** $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$?

Or how to obtain ζ, κ, η ?

$\implies$ use kinetic theory

Or some other microscopic theory

more terms. . .

- Kinetic theory derivation (see Denicol et al., PRD85, 114047 (2012)) or gradient expansion (see Romatschke et al., JHEP 0804, 100 (2008)) lead to even more terms (all possible in second order in products of gradients)
- Do not contribute to entropy, may affect the evolution
- What is usually solved is

$$\pi^{\mu\nu} + \tau_\pi \left[\Delta_\alpha^\mu \Delta_\beta^\nu D\pi^{\alpha\beta} + \frac{4}{3} \pi^{\mu\nu} \nabla_\alpha u^\alpha \right] = \eta \nabla^{\langle\mu} u^{\nu\rangle}$$

$$\eta/s$$

Ideal:

$$(\epsilon + P)Du^\mu = \nabla^\mu P$$

c.f.

$$ma = F$$

$$\eta/s$$

Ideal:

$$(\epsilon + P)Du^\mu = \nabla^\mu P$$

Viscous:

$$\begin{aligned} (\epsilon + P)Du^\mu &= \nabla^\mu P - \Delta^\mu{}_\alpha \partial_\beta \pi^{\alpha\beta} \\ Du^\mu &= \frac{1}{\epsilon + P} \nabla^\mu P - \frac{2\eta}{\epsilon + P} \Delta^\mu{}_\alpha \partial_\beta \left[\nabla^{\langle\alpha} u^{\beta\rangle} + \dots \right] + \dots \end{aligned}$$

$$\mu = 0 \implies Ts = \epsilon + P :$$

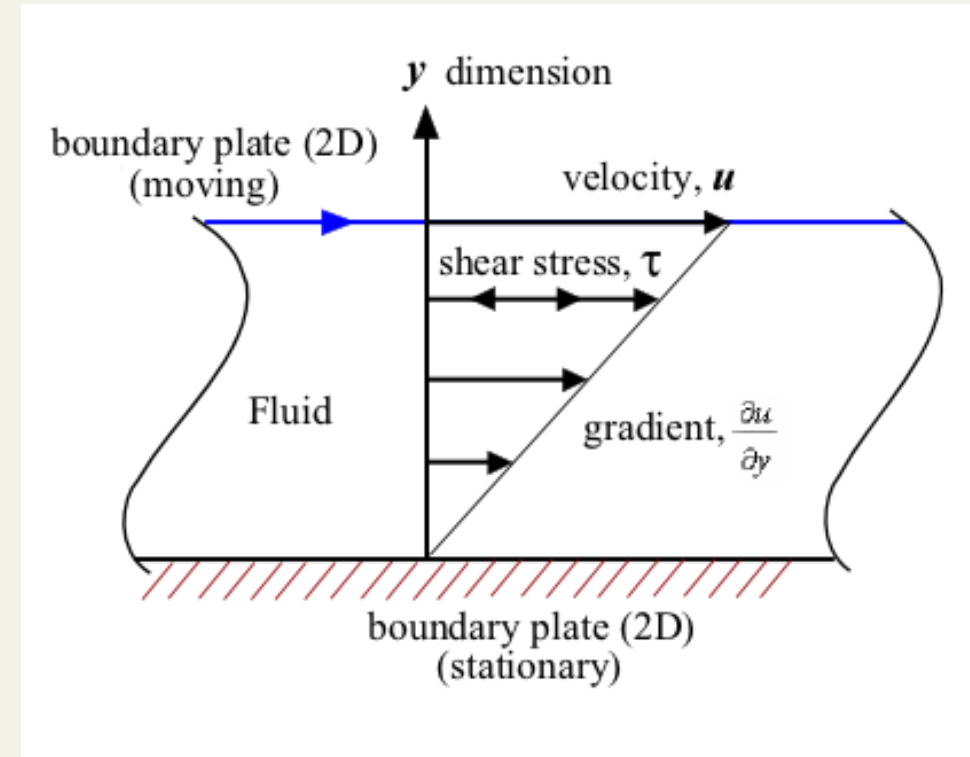
$$Du^\mu = \frac{1}{\epsilon + P} \nabla^\mu P - \frac{2}{T} \frac{\eta}{s} \Delta^\mu{}_\alpha \partial_\beta \left[\nabla^{\langle\alpha} u^{\beta\rangle} + \dots \right] + \dots$$

Shear viscosity

Newton:

$$T_{xy} = -\eta \frac{\partial u_x}{\partial y}$$

acts to reduce velocity gradients



in closed system: energy conserved
kinetic energy gets converted to internal energy
⇒ dissipation

Shear in 1D-bjorken

Navier-Stokes stress

$$\begin{aligned}\pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} &= \text{diag}(0, \frac{2\eta}{3\tau}, \frac{2\eta}{3\tau}, -\frac{4\eta}{3\tau}) \\ T^{\mu\nu} &= \text{diag}(\epsilon, P - \frac{\pi_L}{2}, P - \frac{\pi_L}{2}, P + \pi_L)\end{aligned}$$

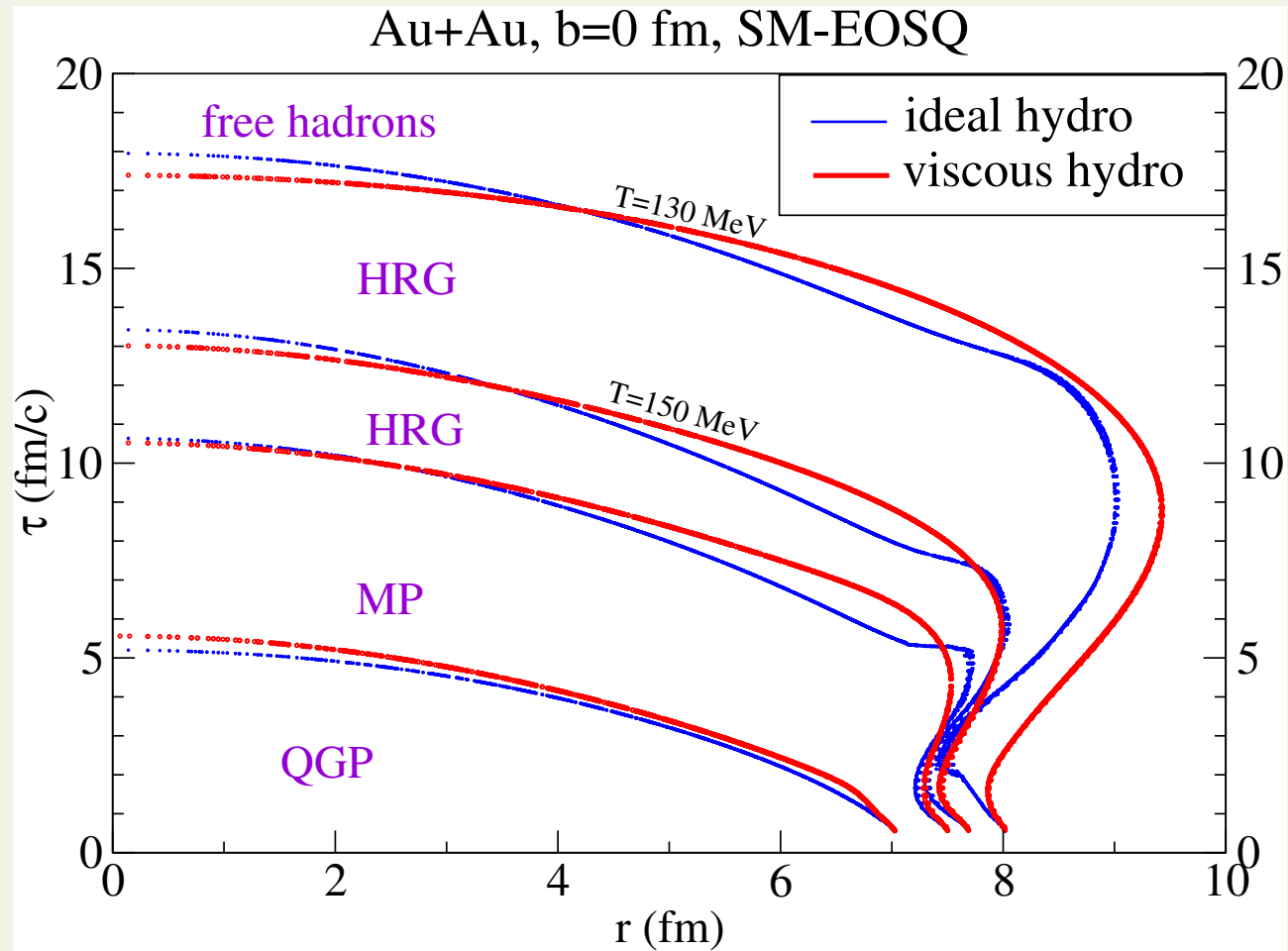
where $\pi_L = \pi^{\eta\eta} = -\frac{4\eta}{3\tau}$

Effective longitudinal pressure $P + \pi_L < P$

Effective transverse pressure $P - \pi_L/2 > P$

Shear **slows down longitudinal** expansion and **accelerates transverse** expansion

Effect on temperature



©Huichao Song

- Edges expand further and stay hotter
- At first core cools slower, later faster

Freeze-out from viscous fluid

Cooper-Frye still works

$$E \frac{dN}{d\mathbf{p}^3} = \int_{\Sigma_{\text{fo}}} d\Sigma_\mu p^\mu f(x, p \cdot u) = \int_{\Sigma_{\text{fo}}} d\Sigma_\mu p^\mu f_0 [1 + \delta f]$$

Grad 14-moment approximation (Boltzmann distribution)

$$\delta f = \varepsilon(x) + \varepsilon_\mu(x) k^\mu + \varepsilon_{\mu\nu} k^\mu k^\nu$$

Shear only, require Landau matching:

$$\begin{aligned} \epsilon &= \int \frac{d^3\mathbf{p}}{E} u_\mu p^\mu u_\nu p^\nu f(x, p \cdot u) = \int \frac{d^3\mathbf{p}}{E} u_\mu p^\mu u_\nu p^\nu f_0 [1 + \delta f] = \epsilon \\ n &= \int \frac{d^3\mathbf{p}}{E} u_\mu p^\mu f(x, p \cdot u) = \int \frac{d^3\mathbf{p}}{E} u_\mu p^\mu f_0 [1 + \delta f] = n \end{aligned}$$

i.e. δf does *not* contribute to ϵ or n

Freeze-out from viscous fluid

Cooper-Frye still works

$$E \frac{dN}{d\mathbf{p}^3} = \int_{\Sigma_{\text{fo}}} d\Sigma_\mu p^\mu f(x, p \cdot u) = \int_{\Sigma_{\text{fo}}} d\Sigma_\mu p^\mu f_0 [1 + \delta f]$$

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$$\delta f = \varepsilon(x) + \varepsilon_\mu(x) k^\mu + \varepsilon_{\mu\nu} k^\mu k^\nu$$

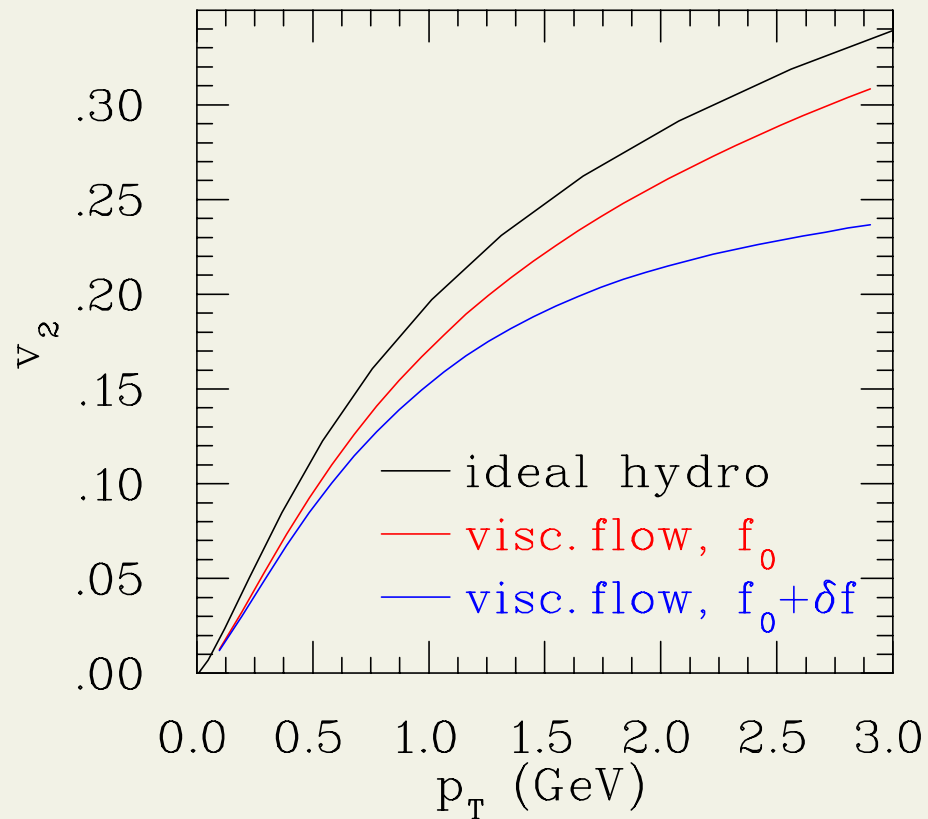
Shear only, Landau matching gives

$$\delta f = \varepsilon_{\mu\nu} k^\mu k^\nu = \frac{1}{2T^2(\epsilon + P)} \pi^{\mu\nu} k_\mu k_\nu$$

Thus, even if velocity and temperature are the same, finite shear causes different particle distributions

How to share $\pi^{\mu\nu}$ for each particle species?

Effect on v_2

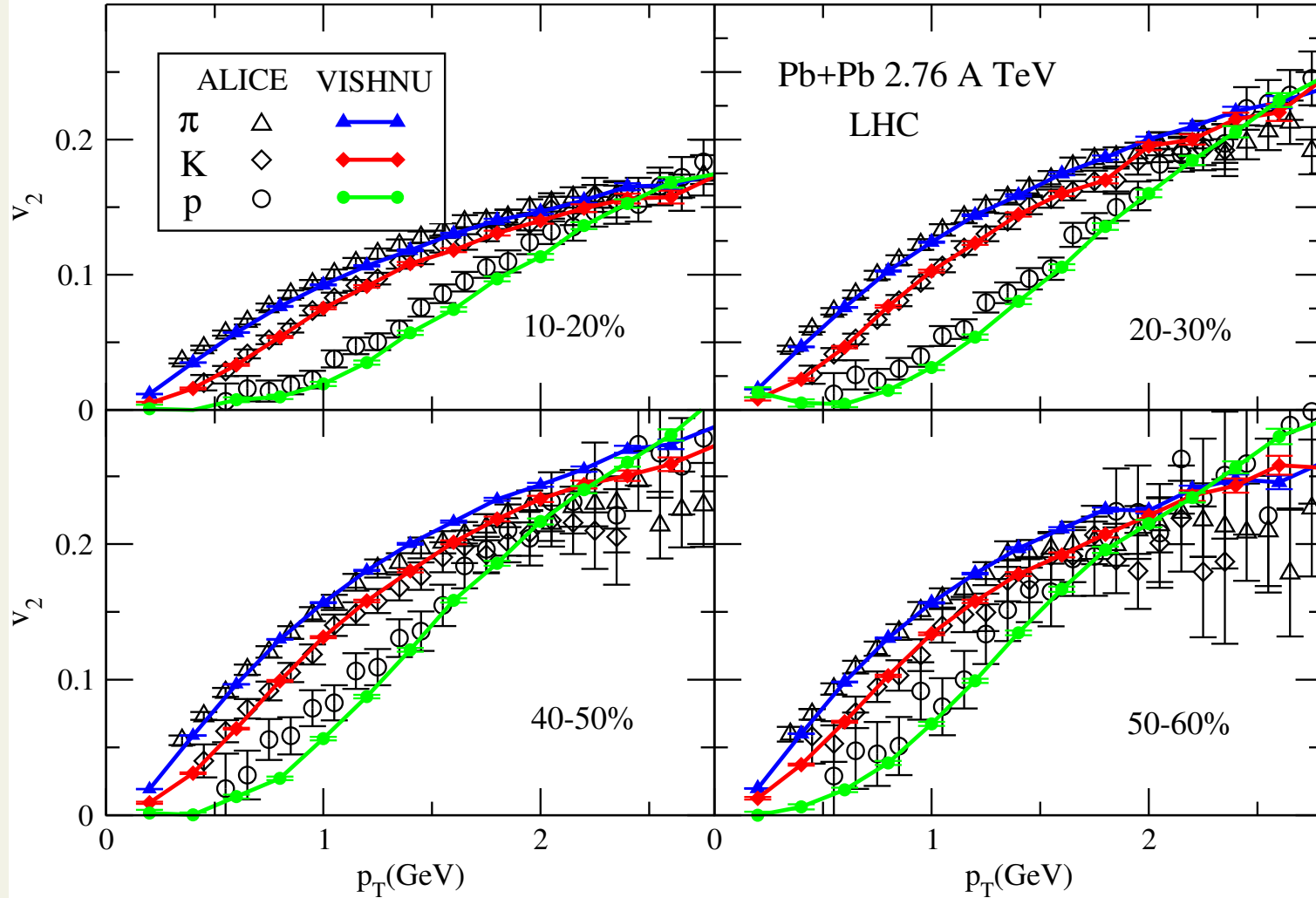


- massless particles
- Note: both **change in flow** and **distributions** affect v_2

$$\eta/s$$

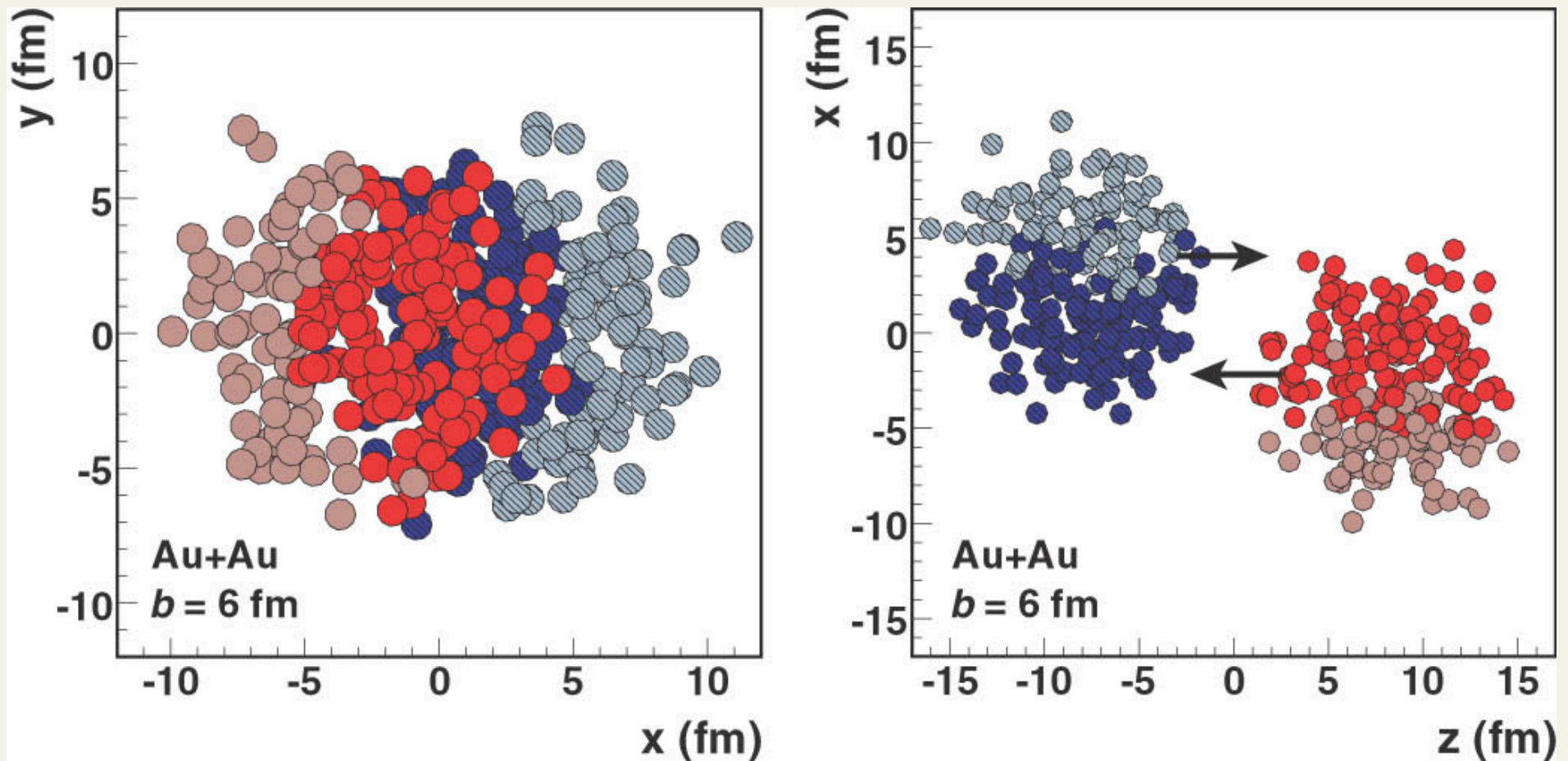
$v_2(p_T)$ of identified particles

Song *et al.* Phys. Rev. C 89, 034919 (2014)



- viscous hydro + UrQMD hybrid
- $\eta/s = 0.16$ favoured

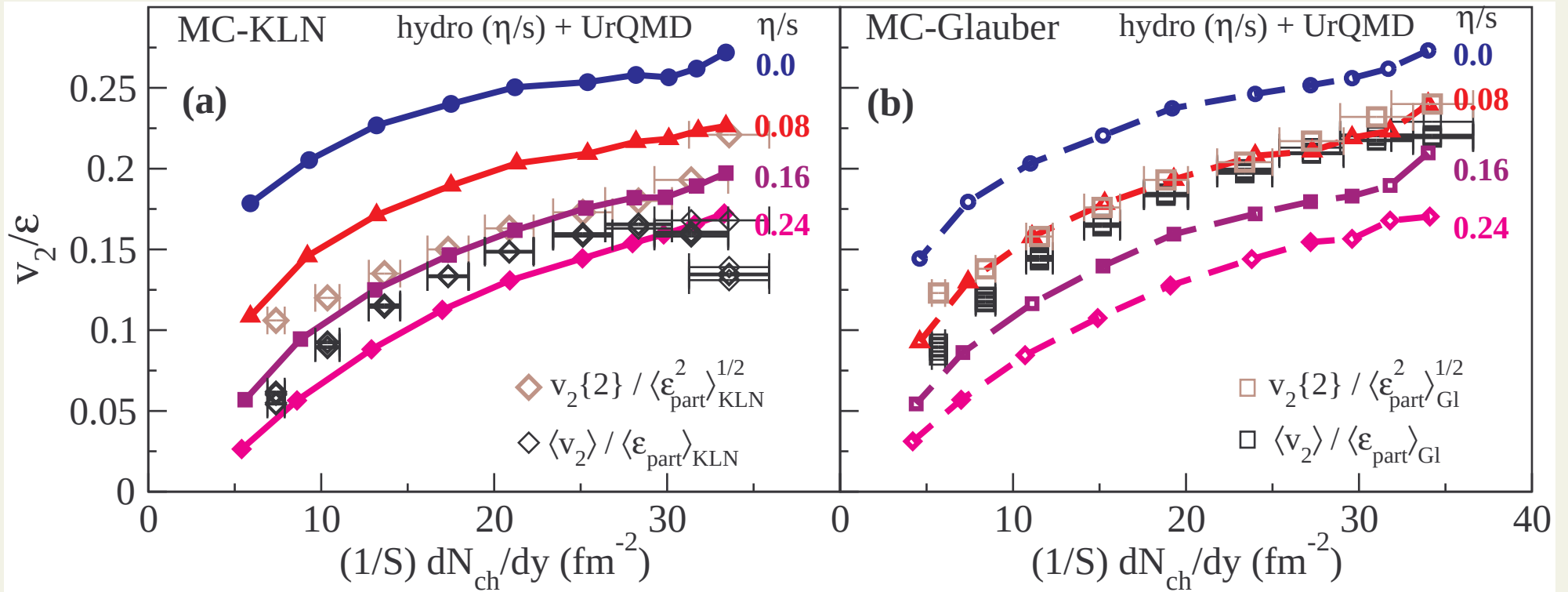
initial shape?



Miller *et al.*, Ann.Rev.Nucl.Part.Sci. 57, 205 (2007)

η/s from v_2

Shen *et al.* J.Phys.G38:124045,2011

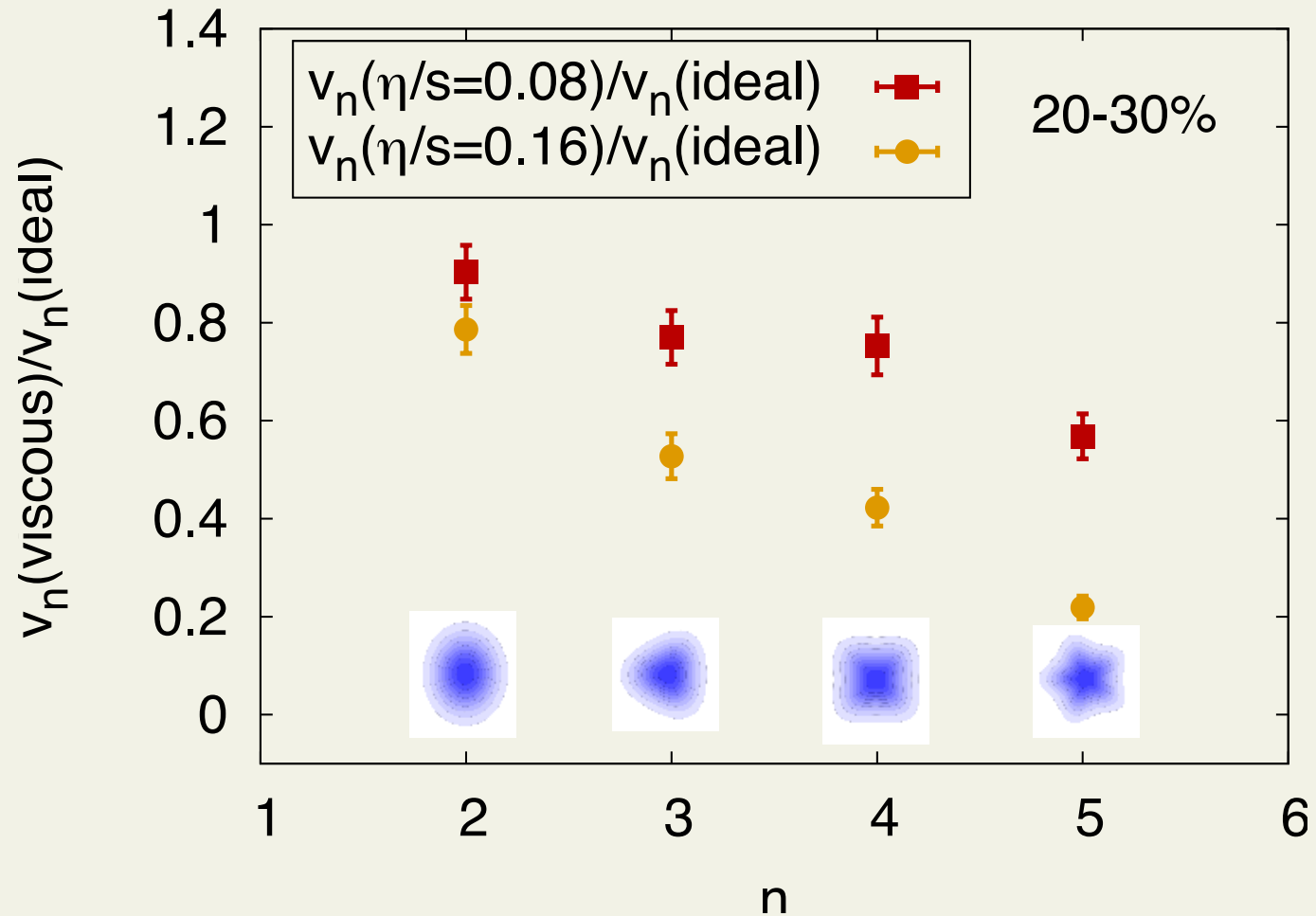


•MC-Glauber initialization: $\eta/s = 0.08$

•MC-KLN initialization: $\eta/s = 0.2$

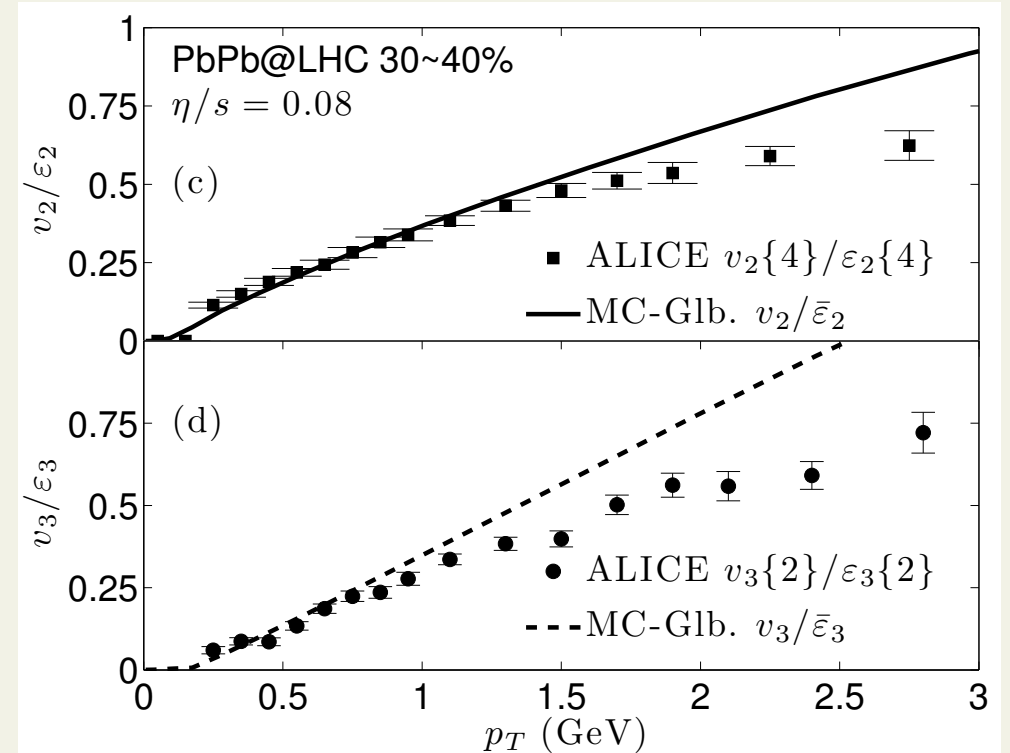
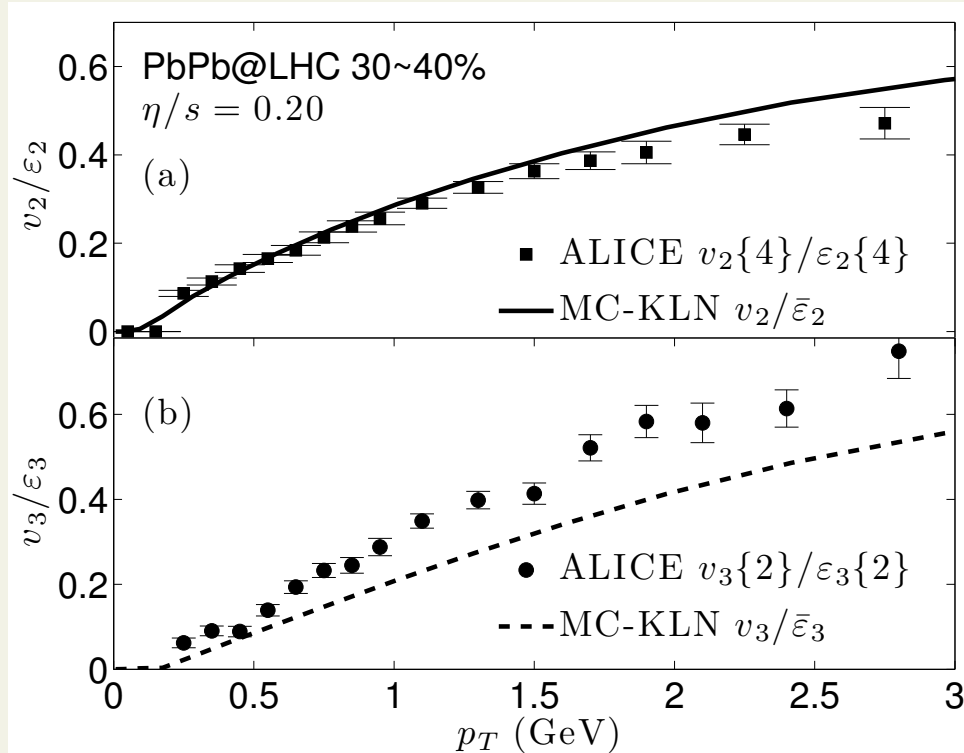
Sensitivity to η/s

Schenke *et al.* Phys.Rev.C85:024901,2012



- higher coefficients are suppressed more by dissipation

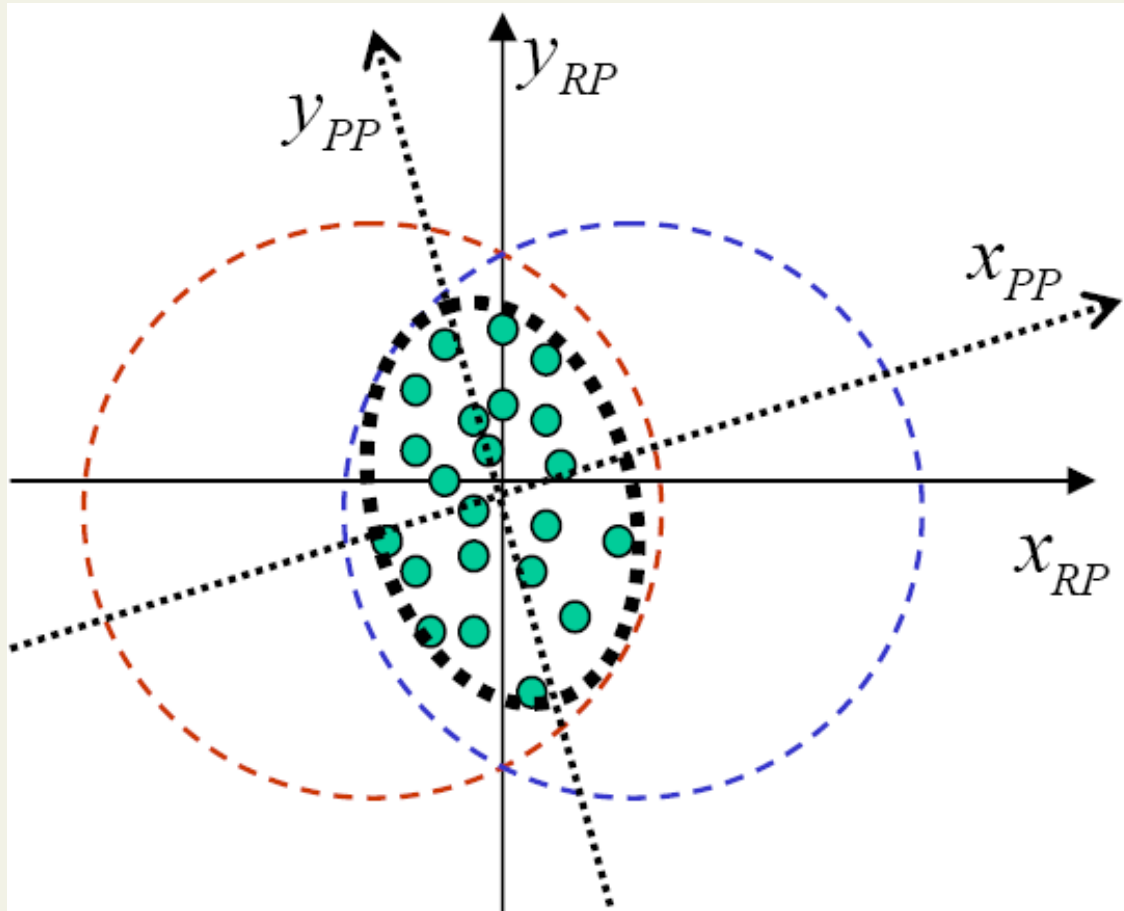
Qiu *et al.* Phys.Lett.B707:151,2012



- models can be distinguished
- MC-Glauber slightly favoured

All the planes. . .

Voloshin et al. Phys. Lett. B 659, 537 (2008)



- X_{RP} : Reaction plane, spanned by beam and impact parameter
- X_{PP} : Participant plane, maximises spatial anisotropy ϵ_n
- Ψ_n : Event plane, maximises anisotropy v_n

Distributions of v_n event-by-event

Scale out the average

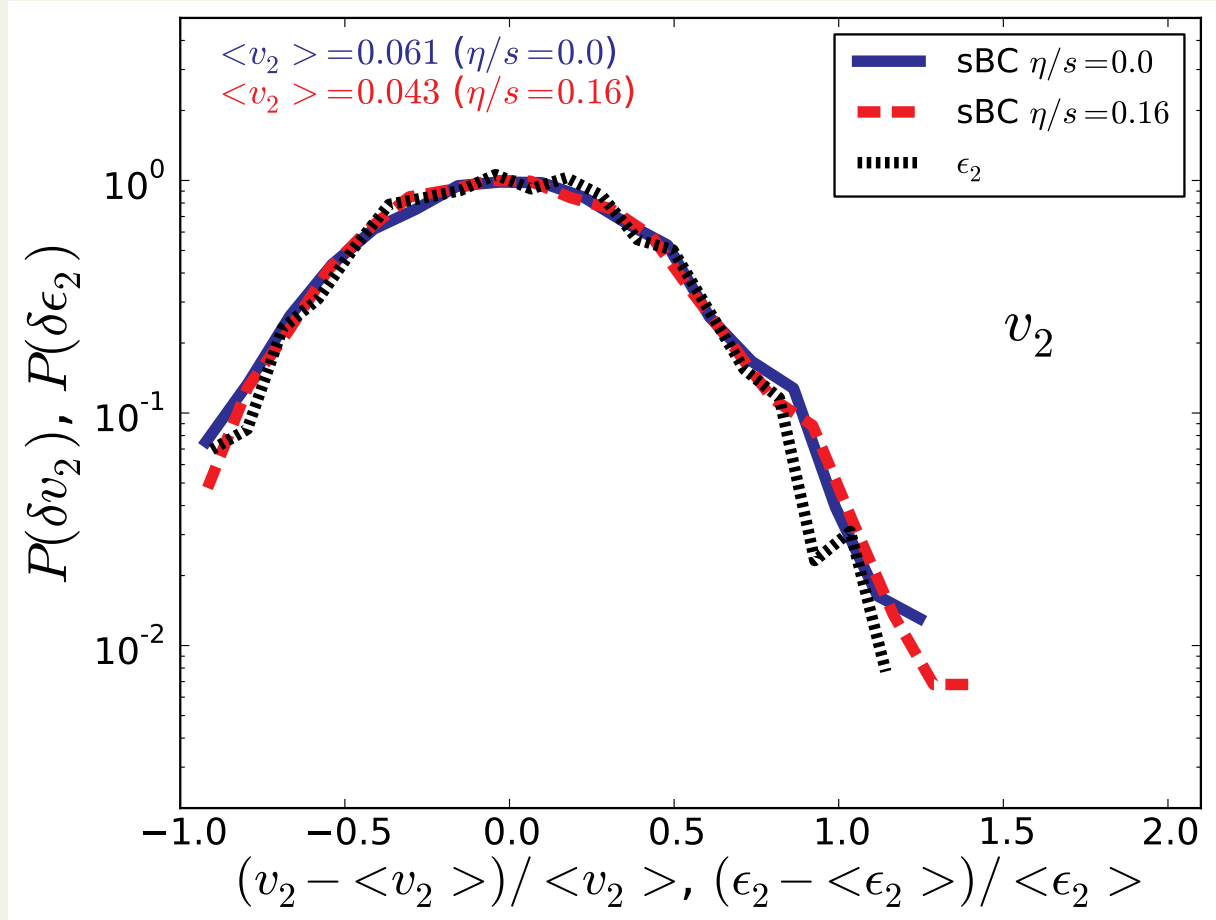
$$\delta v_2 = \frac{v_2 - \langle v_2 \rangle}{\langle v_2 \rangle}$$



$$P(\delta v_2) = P(\delta \epsilon_2)$$

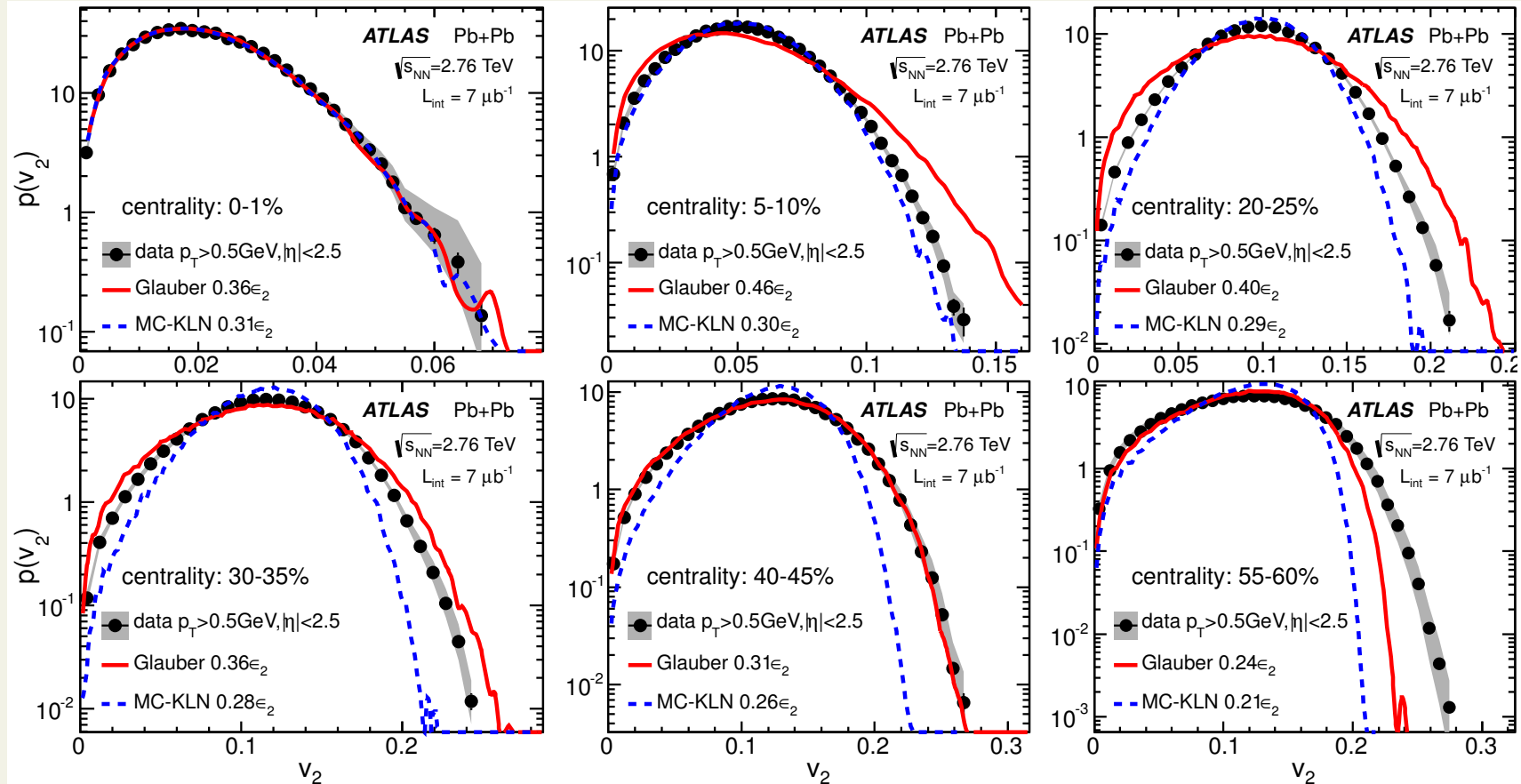
independent of viscosity

Niemi *et al.* Phys.Rev.C87,054901,2013



Flow fluctuations

Aad *et al.* [ATLAS Collaboration] JHEP 1311:183,2013



- $P(v_2)$ compared to MC-Glauber and MC-KLN $P(\varepsilon_2)$
- MC-Glauber initialization: too wide
- MC-KLN initialization: too narrow

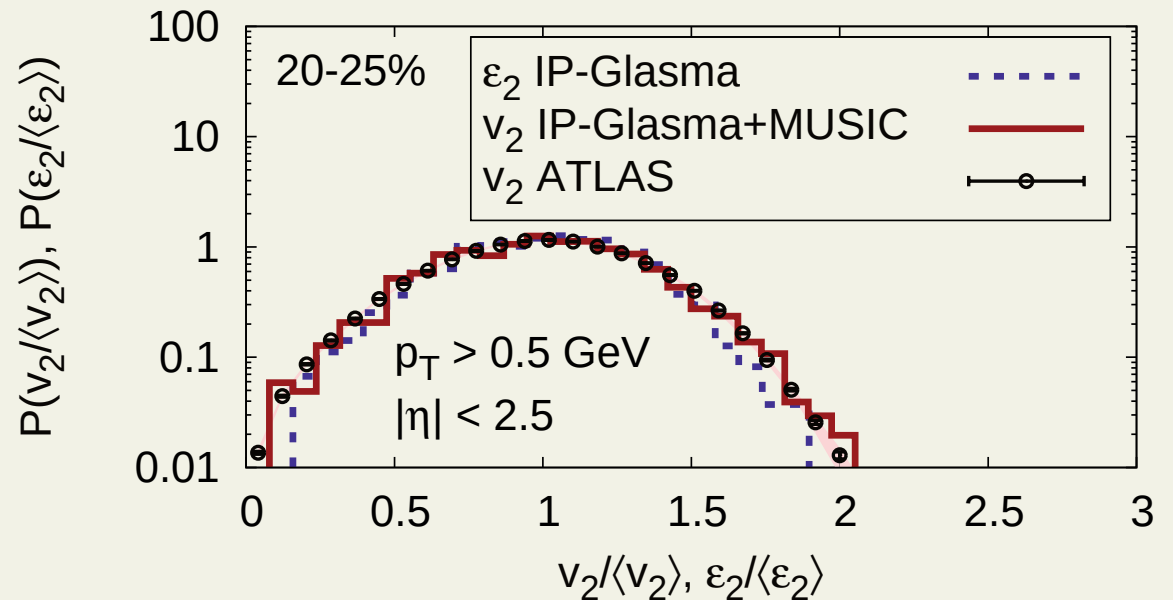
•IP-Glasma
(Color Glass + Yang-Mills)

and

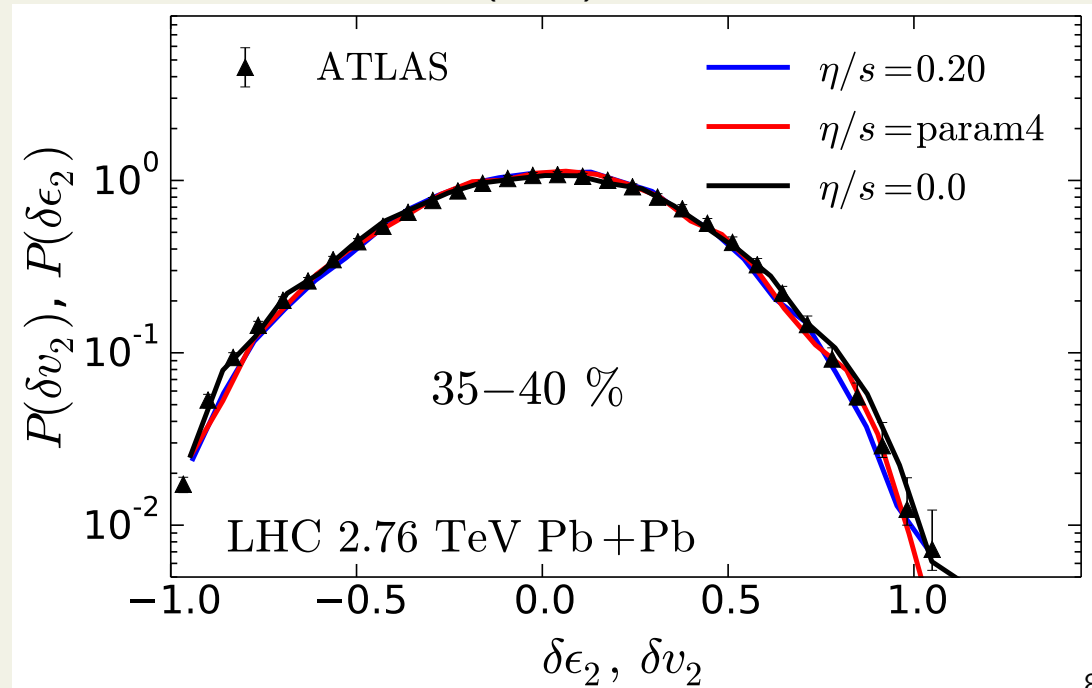
•EKRT
(pQCD + saturation)

initial states work

Gale *et al.*, PRL 110, 012302 (2013)

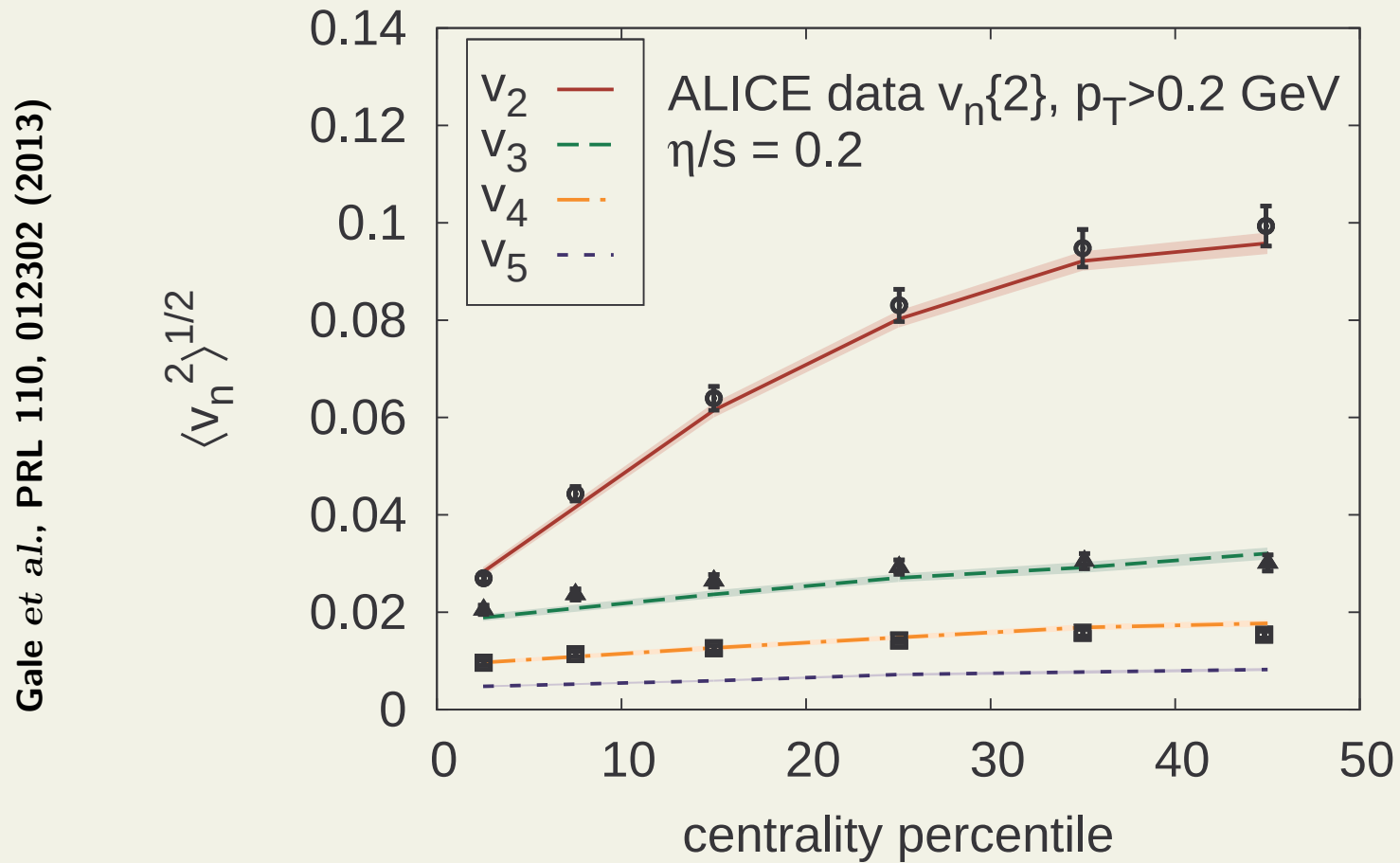


Niemi *et al.*, PRC 93, 024907 (2016)



State of the art

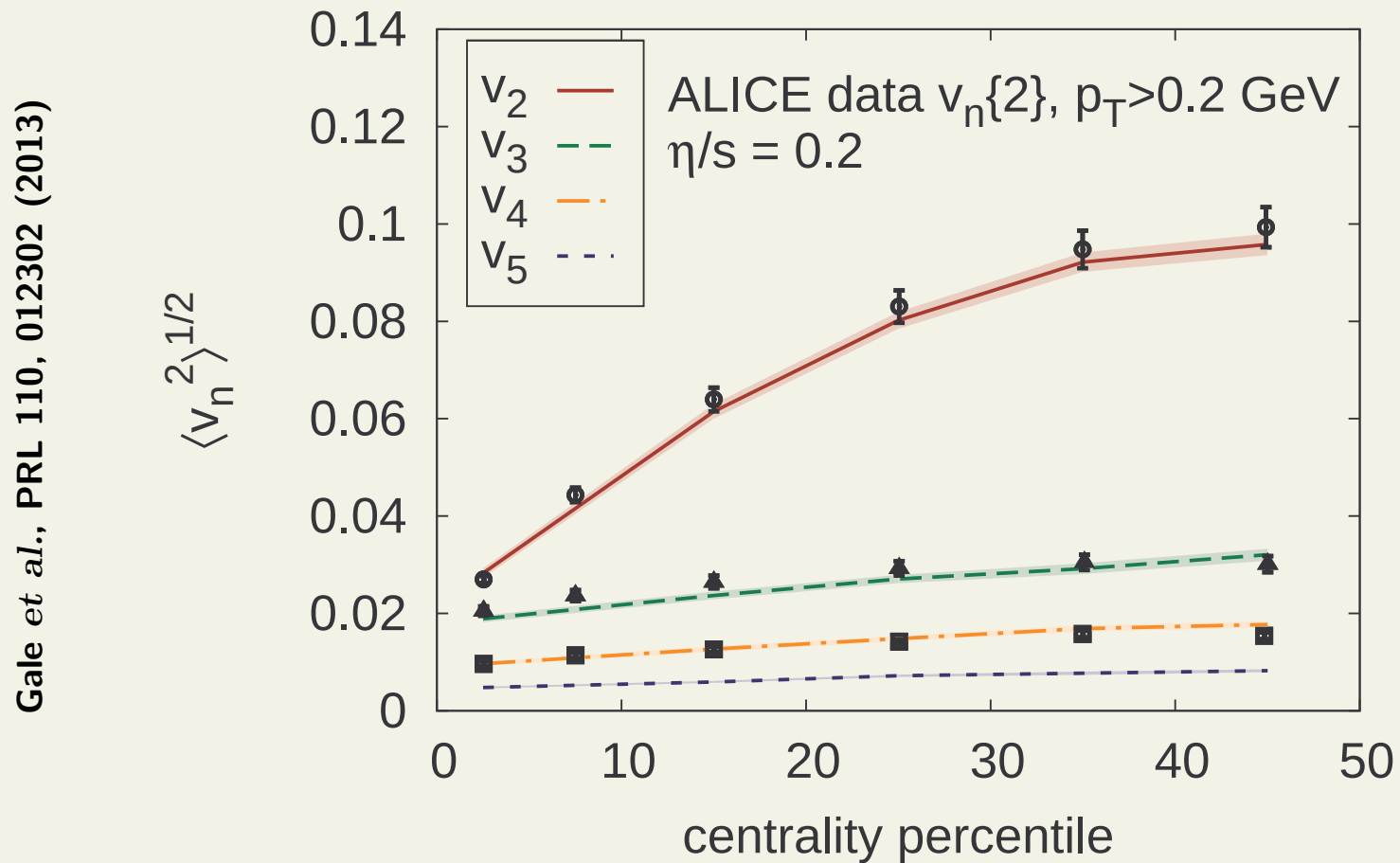
- IP-Glasma initial state: Color Glass plus Yang-Mills evolution



- $\eta/s = 0.2$ favoured

State of the art

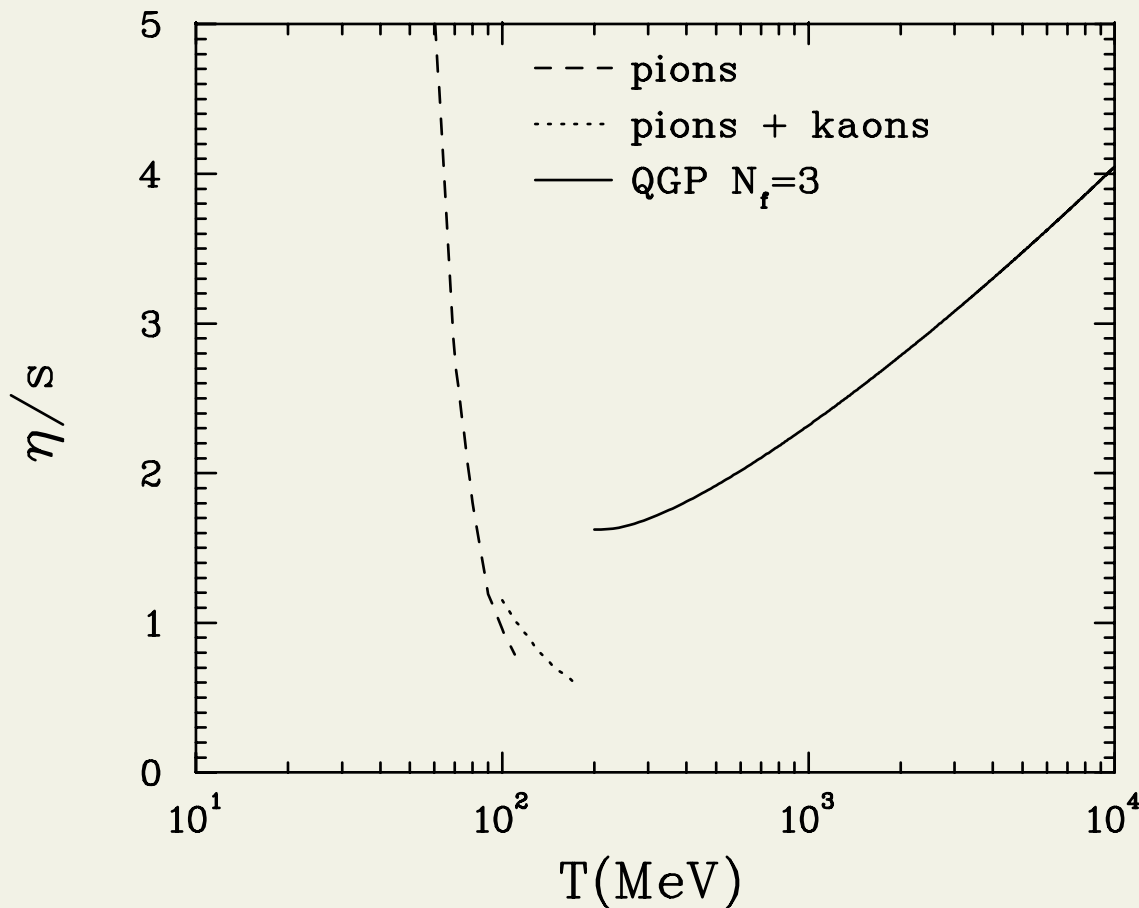
- IP-Glasma initial state: Color Glass plus Yang-Mills evolution



- $\eta/s = 0.2$ favoured **at LHC**
- $\eta/s = 0.12$ favoured **at RHIC**

$$\eta/s = \text{const.} \longrightarrow \eta/s = (\eta/s)(T)$$

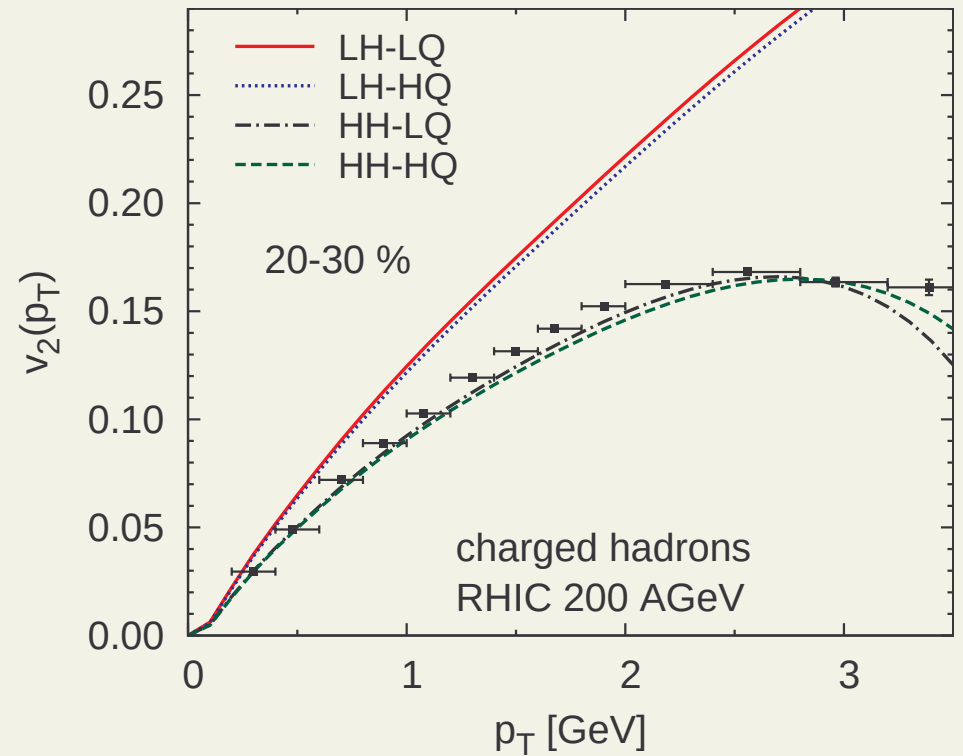
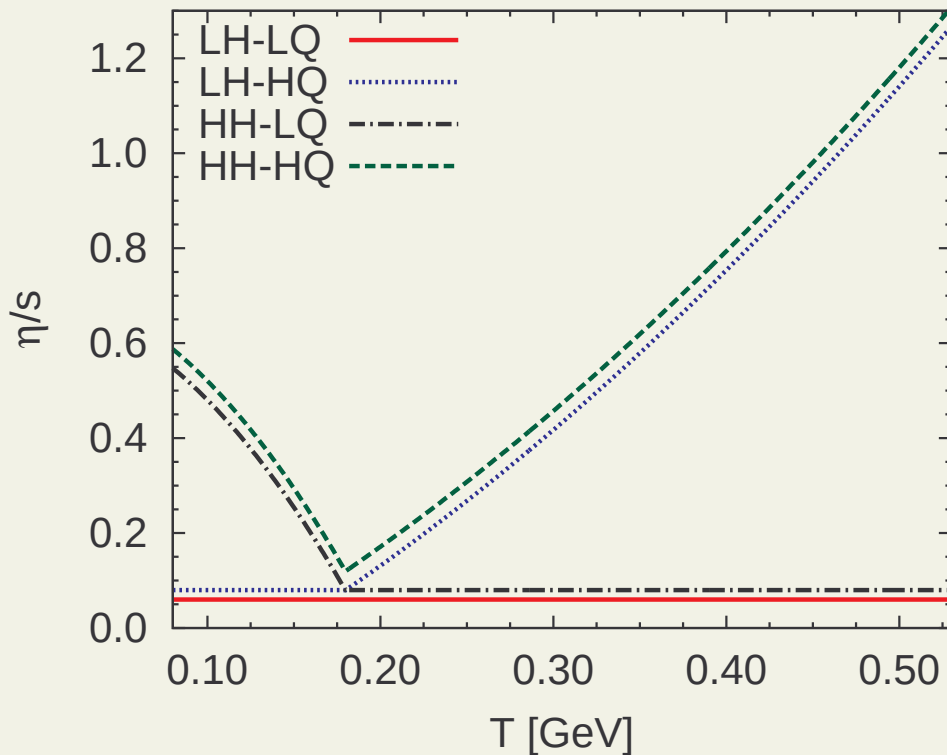
Kapusta, McLerran and Csernai, nucl-th/0604032:



Low T (*Prakash et al.*) using experimental data for 2-body interactions

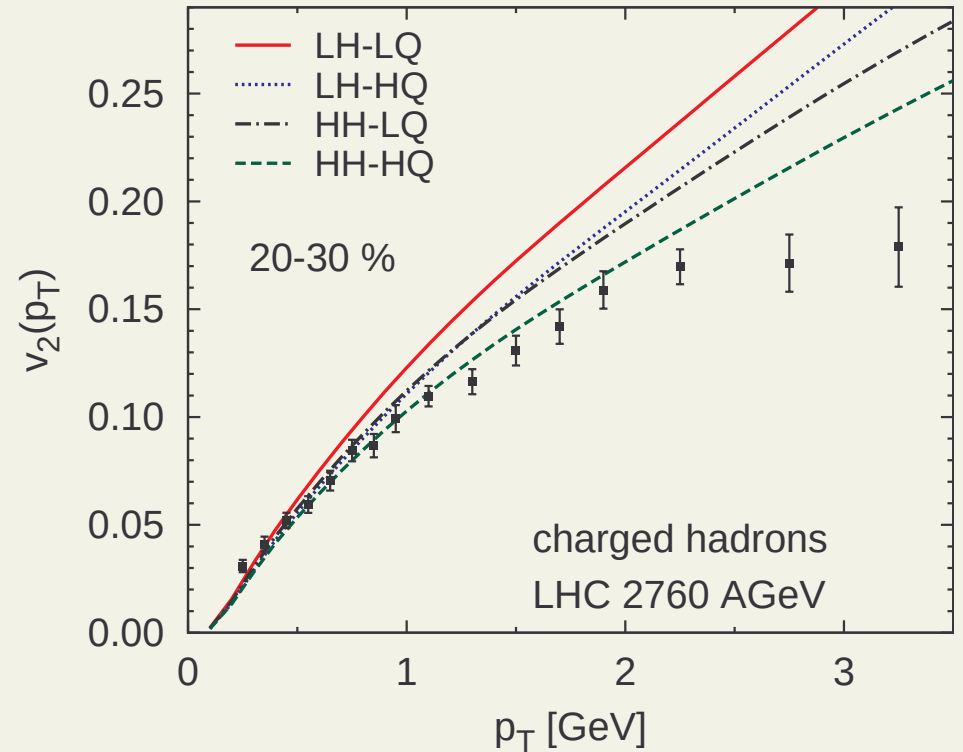
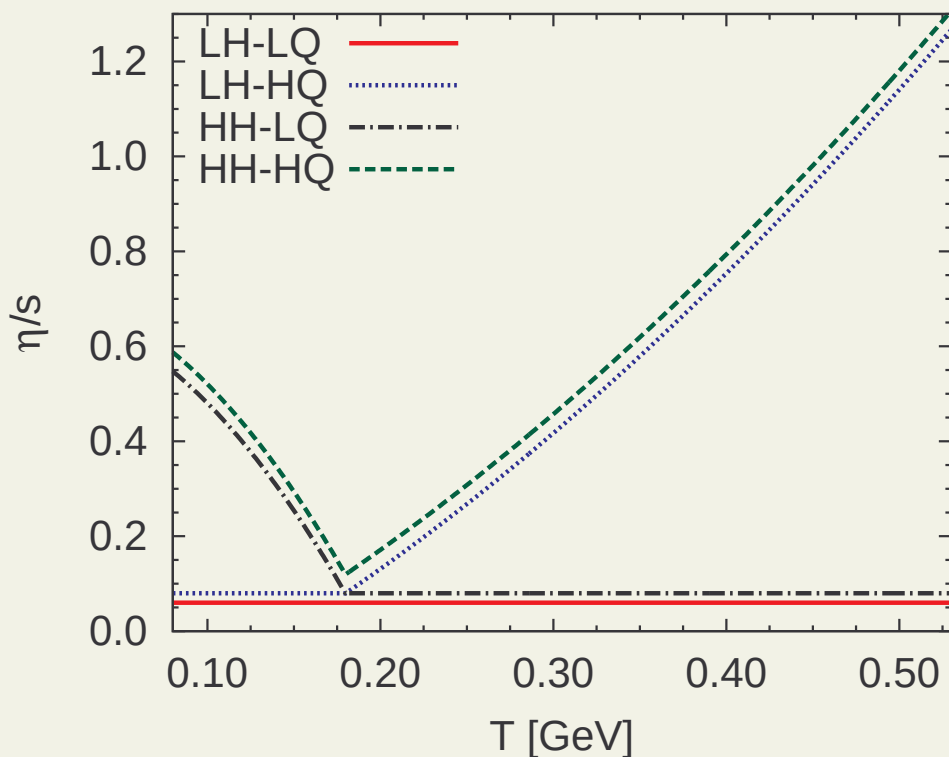
High T (*Yaffe et al.*) using perturbative QCD

$\eta/s(T)$ at RHIC



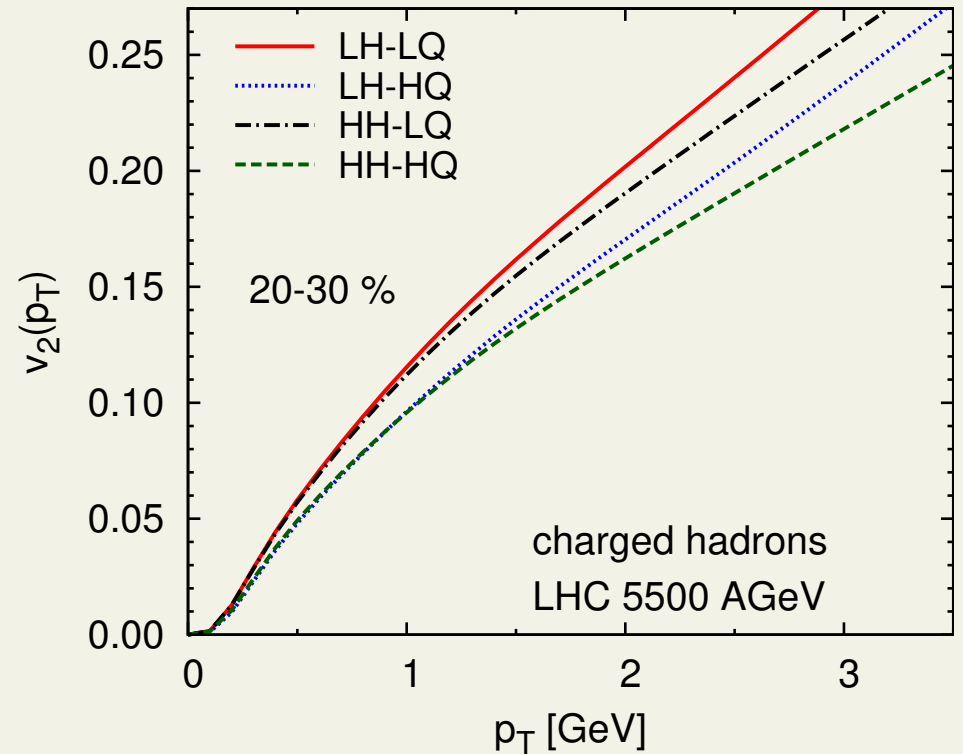
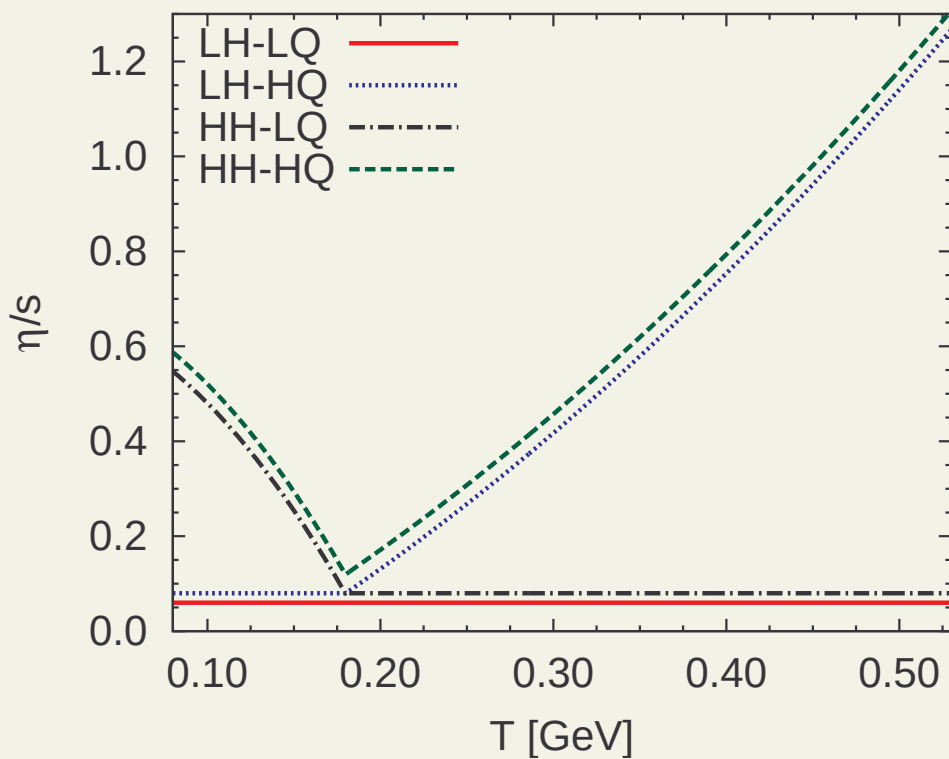
- $v_2(p_T)$ (almost) **insensitive to plasma viscosity!**
- $v_2(p_T)$ **dominated by hadronic viscosity**

$\eta/s(T)$ at LHC ($\sqrt{s_{\text{NN}}} = 2.76 \text{ TeV}$)



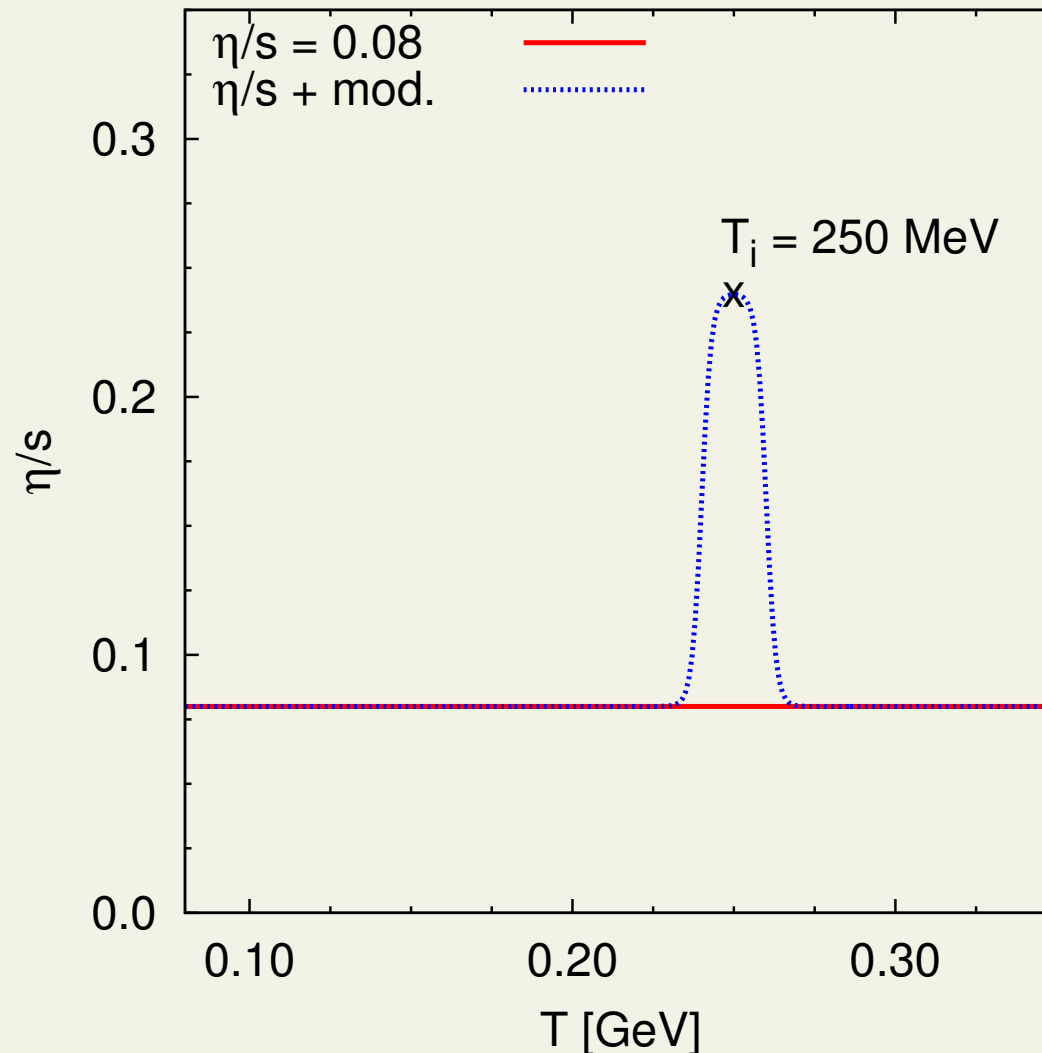
- both **plasma** and **hadronic** η/s affect $v_2(p_T)$

$\eta/s(T)$ at LHC ($\sqrt{s_{\text{NN}}} = 5.5 \text{ TeV}$)



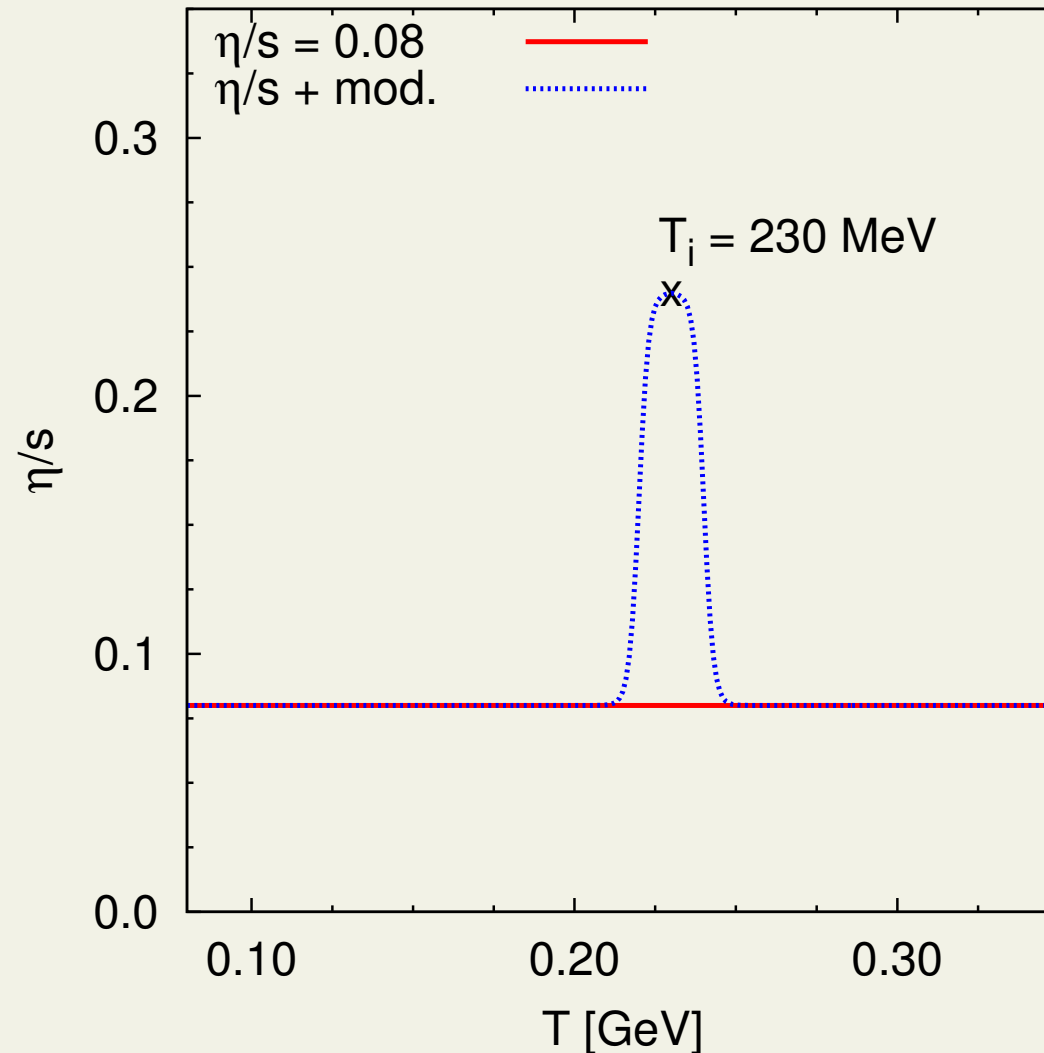
- $v_2(p_T)$ dominated by **plasma viscosity**

At what temperature shear affects v_2 ?



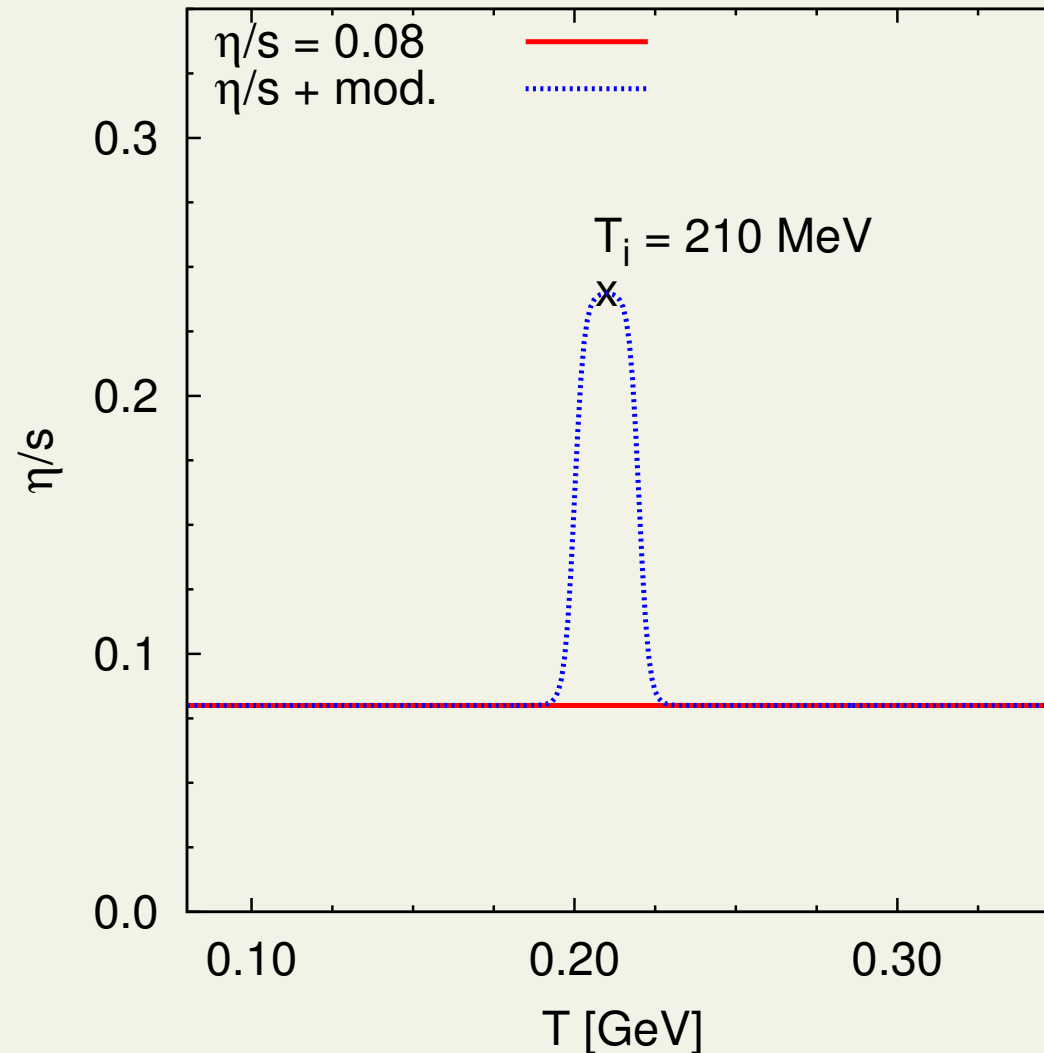
- add peak in η/s at various temperatures

At what temperature shear affects v_2 ?



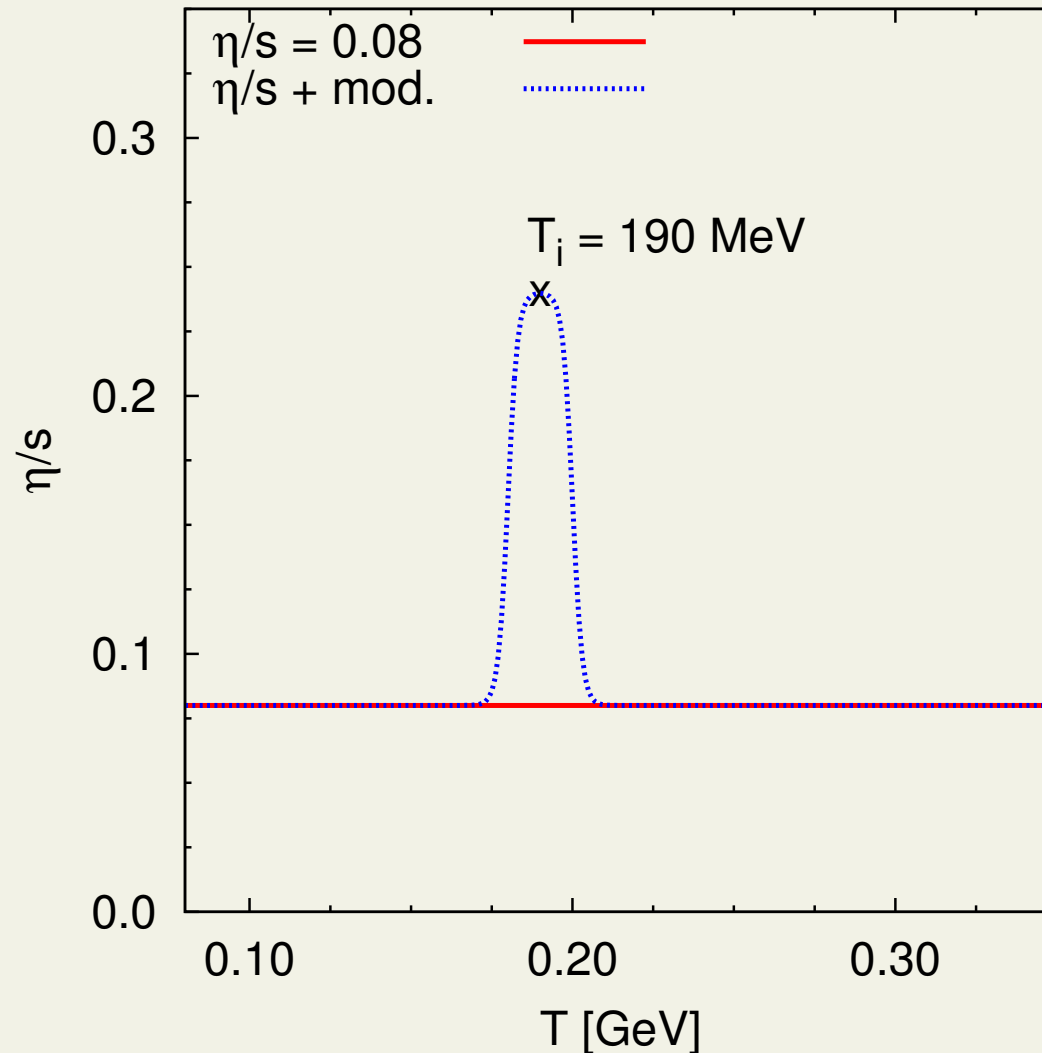
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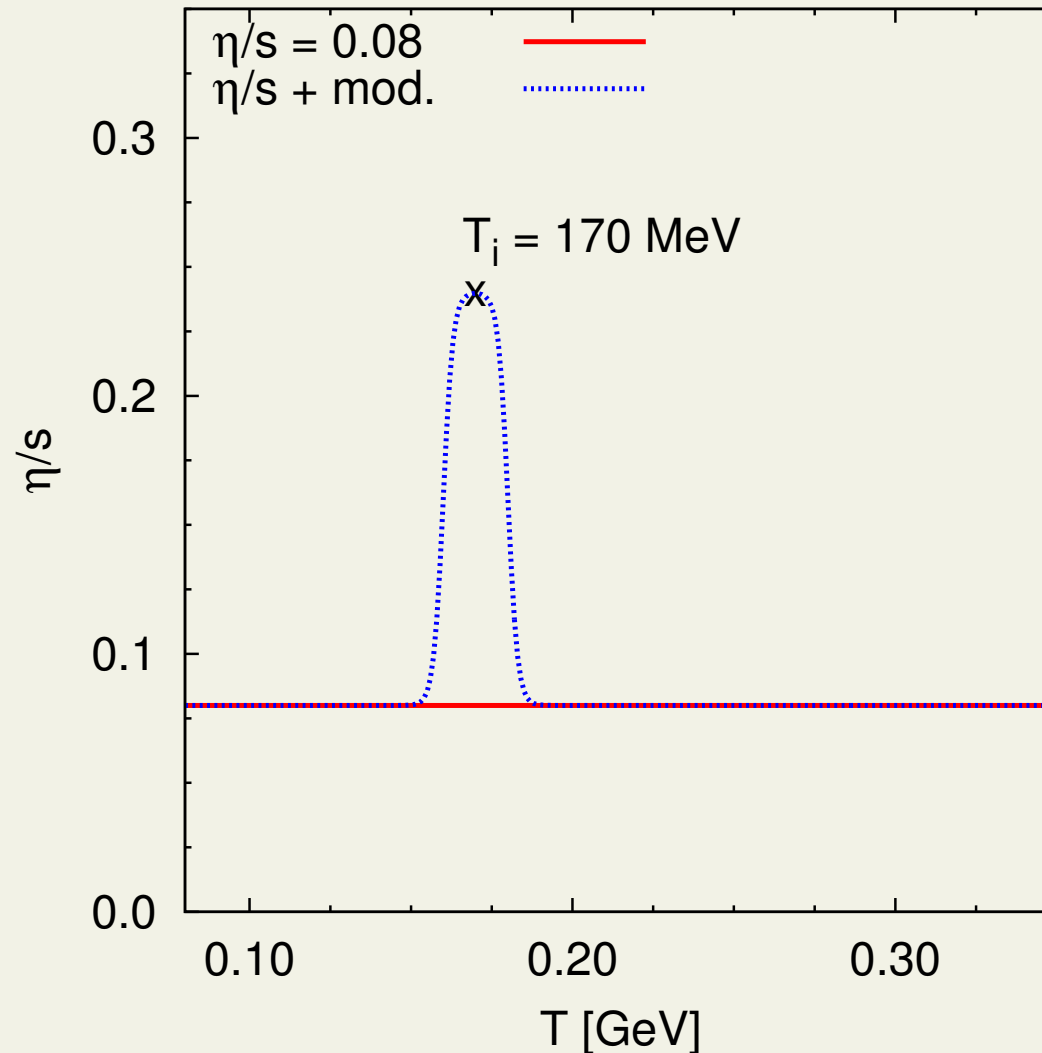
- add peak in η/s at various temperatures

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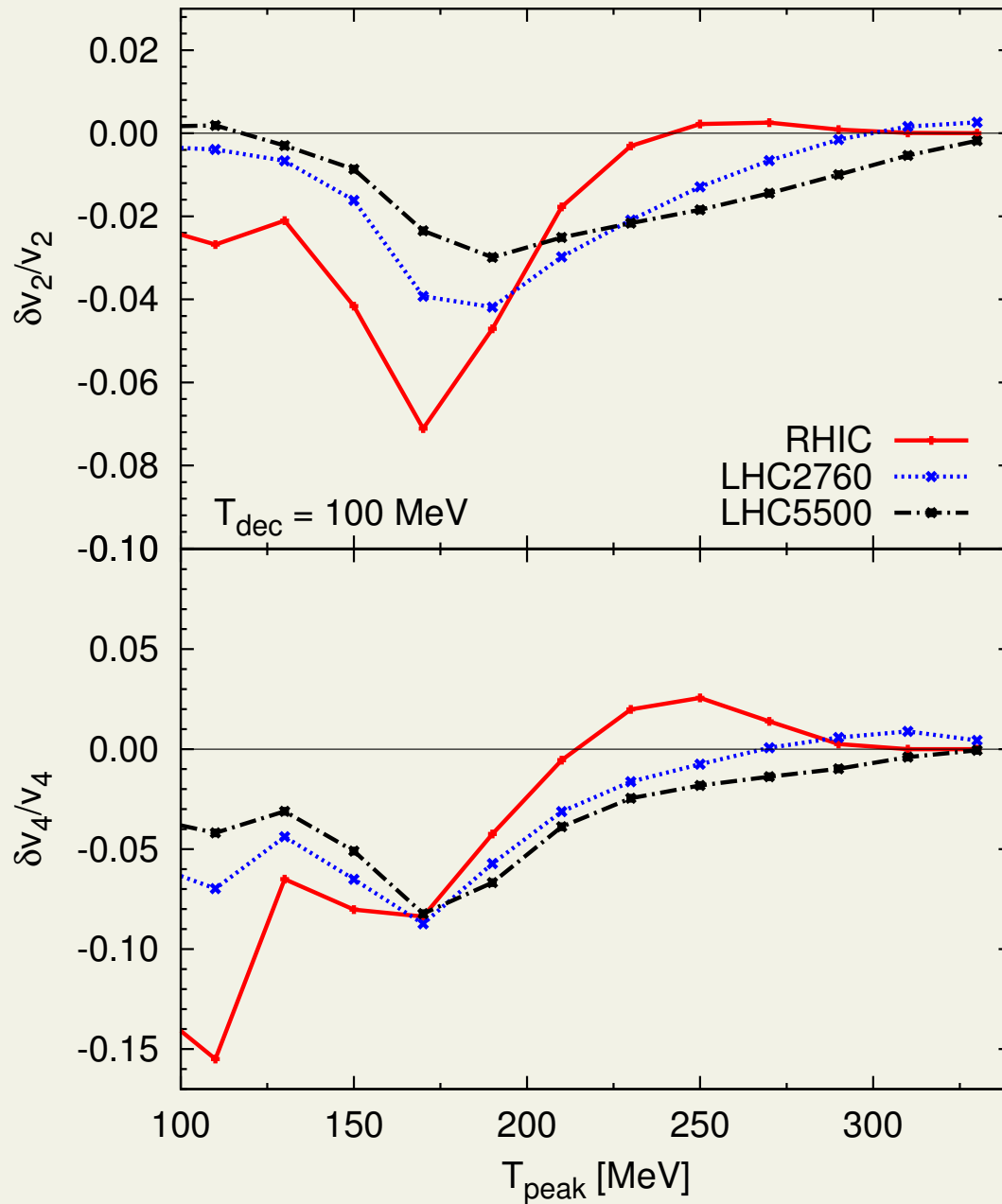
- add peak in η/s at various temperatures

At what temperature shear affects v_2 ?



- add peak in η/s at various temperatures

Sensitivity to shear



- 20-30% centrality

$$\delta v_2 = v_2(\eta/s(T)) - v_2(\eta/s = 0.08)$$

- sensitive region moves to larger temperature and becomes wider with increasing collision energy

How come?

Shear **slows down longitudinal** expansion and **accelerates transverse** expansion

⇒ larger transverse flow v_r

Shear **reduces** azimuthal anisotropy

⇒ smaller anisotropy δv

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⇒ larger transverse flow v_r

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- the larger the v_r the larger the v_2
- the larger the δv the larger the v_2

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Thus at early times shear both **increases** and **decreases** v_2 !

⇒ net effect zero

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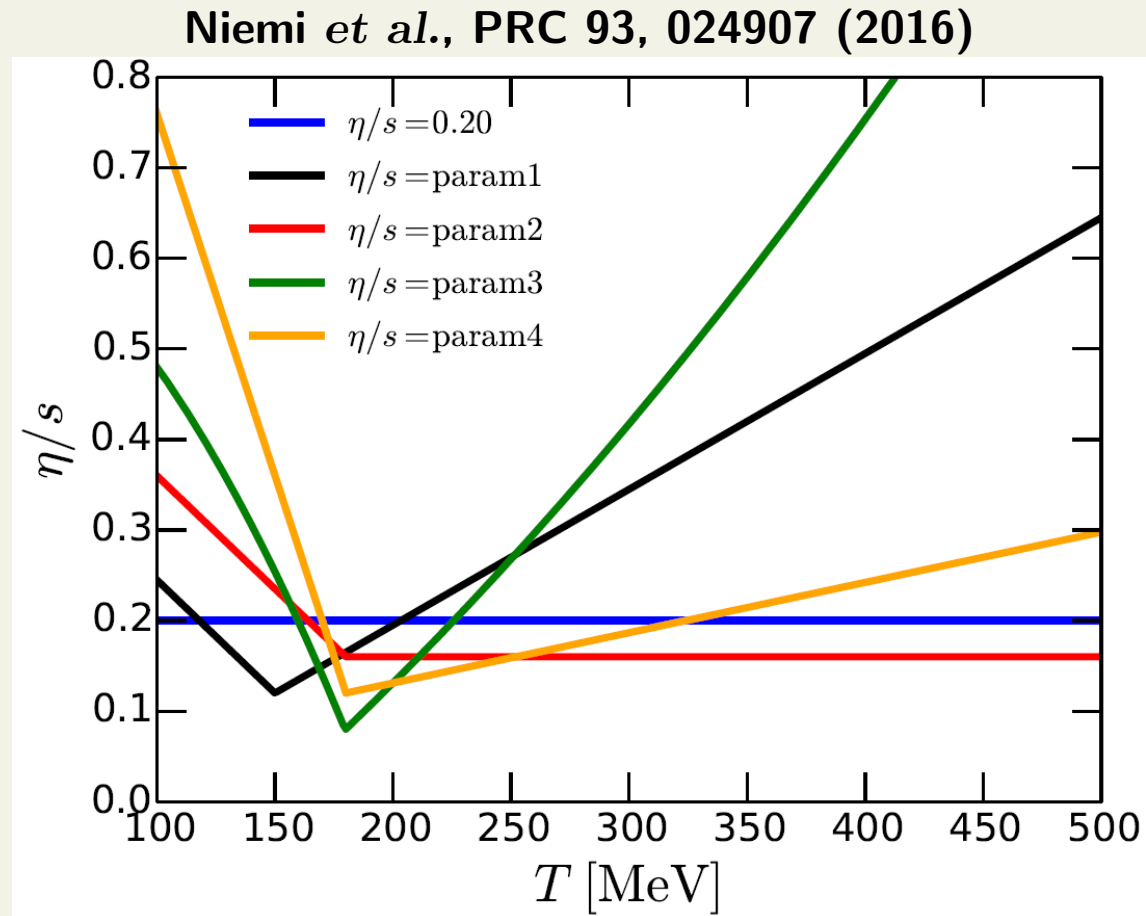
Thus at early times shear both **increases** and **decreases** v_2 !

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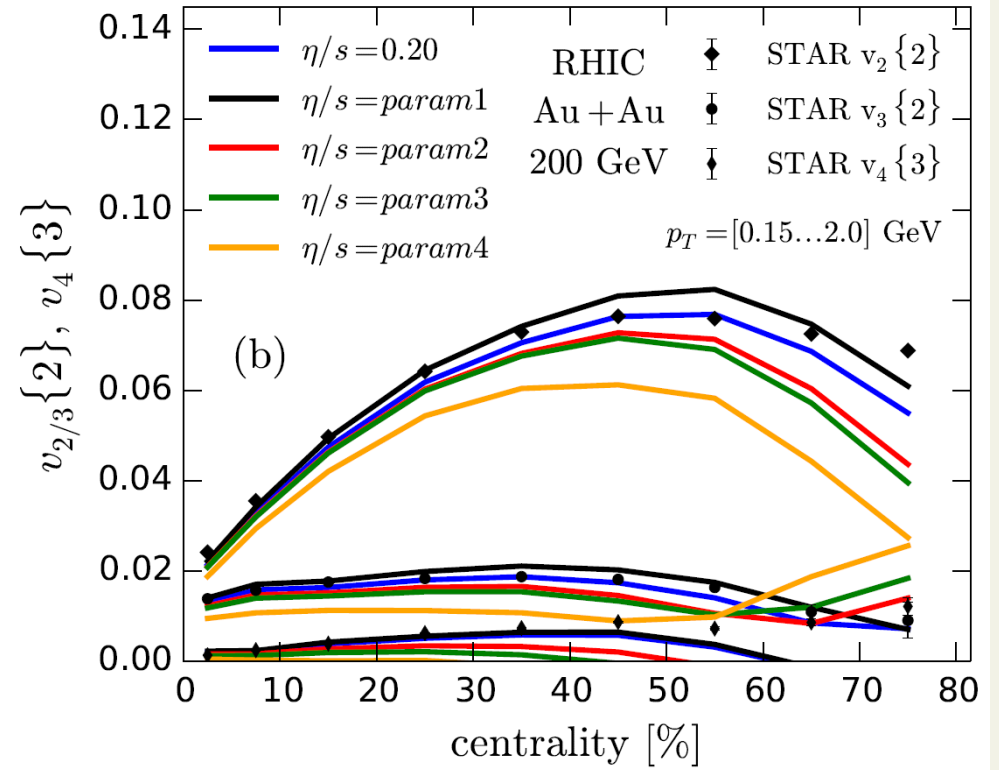
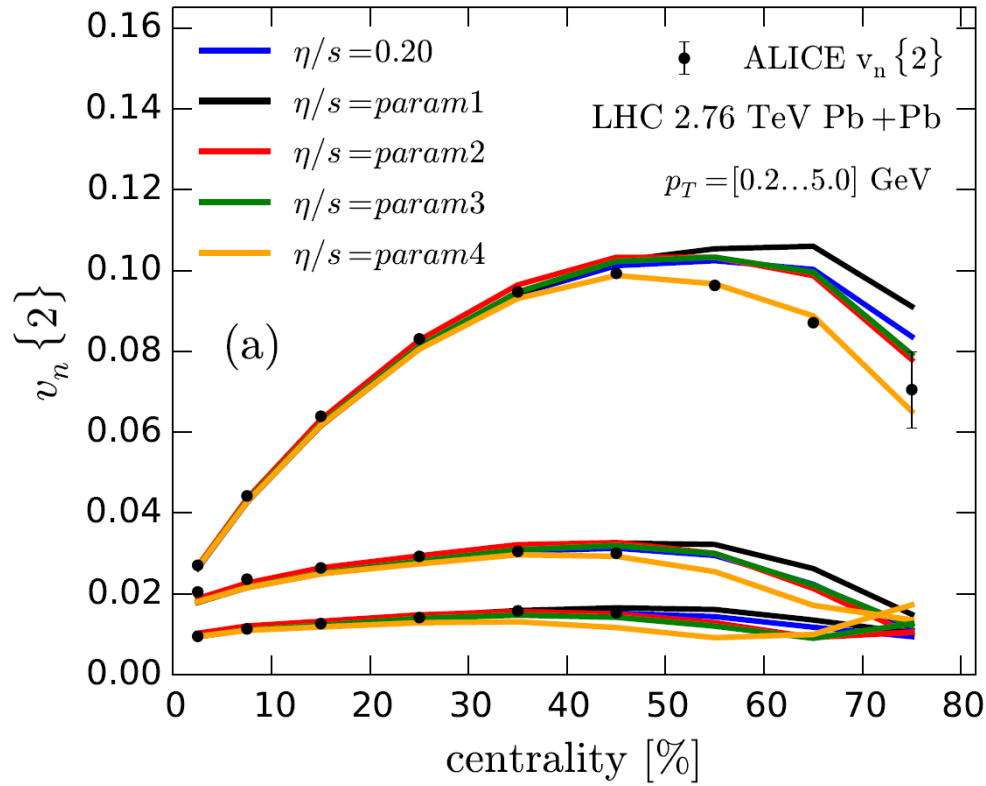
Shear reduces v_2 only after **transverse flow has built up**

- at RHIC this happens around hadronisation
- at LHC this happens before hadronisation

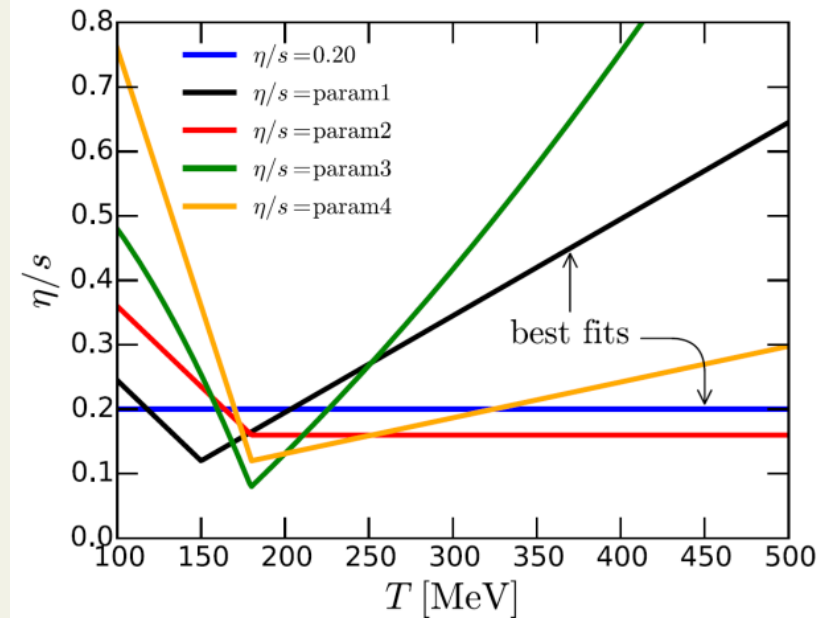
$$(\eta/s)(T)$$



- family of η/s parametrisations
- constrained to reproduce v_2 at LHC



Low η/s in HRG favoured



Summary

- Hydrodynamics is a useful tool to model collision dynamics
 - approximation at its best
 - but it can reproduce (some of) the data
- we have observed hydrodynamical behaviour at RHIC and LHC

Literature:

Textbook:

- W. Florkowski,
Phenomenology of Ultra-relativistic Heavy-ion Collisions, (World-Scientific, 2010)

Reviews:

- P. F. Kolb and U. Heinz, in *Quark Gluon Plasma 3* [nucl-th/0305084]
- P. Huovinen, in *Quark Gluon Plasma 3* [nucl-th/0305064]
- P. Huovinen and P. V. Ruuskanen, *Ann. Rev. Nucl. Part. Sci.* 56, 163 (2006)
- D. Teaney, in *Quark Gluon Plasma 4*, arXiv:0905.2433 [nucl-th]
- P. Romatschke, *Int. J. Mod. Phys. E* 19, 1 (2010)
- C. Gale, S. Jeon and B. Schenke, *Int. J. Mod. Phys. A* 28, 1340011 (2013)
- U. W. Heinz and R. Snellings, arXiv:1301.2826 [nucl-th]

Hydrodynamics lecture notes:

- D. H. Rischke, nucl-th/9809044

Dissipative hydro from kinetic theory:

- S. R. de Groot, W. A. van Leeuwen, and Ch. G. van Weert,
Relativistic kinetic theory – Principles and applications, (North-Holland, 1980)
- G. S. Denicol, H. Niemi, E. Molnar and D. H. Rischke, *Phys. Rev. D* 85, 114047 (2012)