A computational algorithm for covariant series expansions in general relativity

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References

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Normal neighborhood of a submanifold

A real analytic manifold H, $\dim H = n+m$, $n > m \geqslant 0$

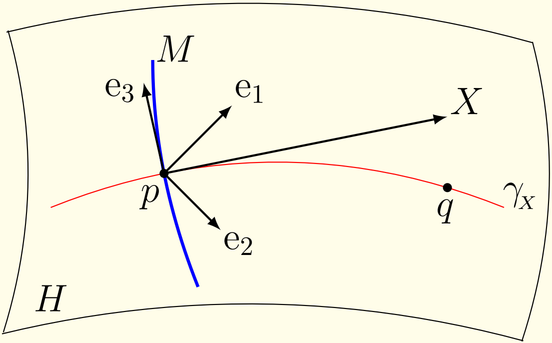
A connected parallelizable submanifold M, $\dim M = m$
If $\dim M = m = 0$ then M is a point on H

$p \in M$, $T_p(H) = T_p(M) \oplus O_p$, $X = X^i \mathrm{e}_i = X^\alpha \mathrm{e}_\alpha + X^a \mathrm{e}_a$

The Latin indices $i, j, \ldots$ run from 1 to $n + m$: $T_p(H)$
The Greek indices $\alpha, \beta, \ldots$ run from 1 to n: O_p
The Latin indices $a, b, \ldots$ run from $m+1$ to $n+m$: $T_p(M)$

A linear connection ∇ on H: $\nabla_k = \nabla_{e_k}$, $\nabla_i e_j = \Gamma^k_{ij} e_k$

The exponential map: $\gamma_X(0)=p,\ \gamma_X(1)=q$



$p \in M\ ,\ \ X \in O_p$
$O_p = \langle \mathrm{e}_1, \mathrm{e}_2 \rangle$
$T_p(M) = \langle \mathrm{e}_3 \rangle$

γ_X is a geodesic on H in the direction of X starting at $p \in M$

At each point $p \in M$ we choose a basis $\{\mathrm{e}_i\}_1^{m+n}$ such that the vectors $\{\mathrm{e}_a\}_{m+1}^{m+n}$ are tangent to M and the vectors $\{\mathrm{e}_a\}_1^{n}$ are transverse to M

General form of the covariant expansion I

$$\left(Q^{j_1\dots j_s}_{i_1\dots i_r}\right)_q = \sum_{\sigma+|\mu|+|\nu|\geq 0} \frac{1}{\sigma!} X^{\gamma_1}\dots X^{\gamma_\sigma}\left(Q^{l_1\dots l_s}_{k_1\dots k_r;\gamma_1\dots\gamma_\sigma}\right)_p \times$$

$$\times \underset{(\mu_1)}{u}{}^{k_1}_{i_1} \dots \underset{(\mu_r)}{u}{}^{k_r}_{i_r} \underset{(\nu_1)}{v}{}^{j_1}_{l_1} \dots \underset{(\nu_s)}{u}{}^{k_s}_{i_s},$$

$$\sigma, \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s \geq 0, \quad |\mu| = \mu_1 + \dots + \mu_r, \quad |\nu| = \nu_1 + \dots + \nu_s,$$

$$\underset{(\mu)}{v}{}^{k}_{i} = -\sum_{\sigma=1}^{\mu} \underset{(\mu-\sigma)}{v}{}^{k}_{l} \underset{(\sigma)}{u}{}^{l}_{i}, \quad \mu \geq 1.$$

$$\underset{(0)}{u}{}^{k}_{i} = \underset{(0)}{v}{}^{k}_{i} = \delta^k_i, \quad \underset{(1)}{u}{}^{k}_{\alpha} = \frac{1}{2} X^\beta (T^k_{\beta\alpha})_p, \quad \underset{(1)}{u}{}^{k}_{a} = X^\beta(\Gamma^k_{\beta a} + T^k_{\beta a})_p,$$

General form of the covariant expansion II

$$\underset{(\mu)}{u}{}_i^{\,k} =$$

$$\frac{2}{2\mu+\varepsilon(i)+1}\sum_{\sigma=1}^{\mu}\frac{1}{(\sigma-1)!}X^{\alpha_1}\dots X^{\alpha_\sigma}\left(T^k_{\alpha_1 l\,;\,\alpha_2\dots\alpha_\sigma}\right)_p \underset{(\mu-\sigma)}{u}{}_i^{\,l} +$$

$$\frac{1}{\mu(\mu+\varepsilon(i))}\sum_{\sigma=2}^{\mu}\frac{1}{(\sigma-2)!}X^{\alpha_1}\dots X^{\alpha_\sigma}\left(R^k_{\alpha_1\alpha_2 l\,;\,\alpha_3\dots\alpha_\sigma}\right)_p \underset{(\mu-\sigma)}{u}{}_i^{\,l} \,,$$

$$\mu \geq 2\,, \quad \varepsilon(\alpha) = 1\,, \quad \varepsilon(a) = -1\,,$$

The expansion of a Riemannian metric

$$(g_{ik})_q = \sum_{\mu+\nu\geq 0} (g_{jl})_p \underset{(\mu)}{u}{}^j_i \underset{(\nu)}{u}{}^l_k \,,$$

$$\underset{(0)}{u}{}^k_i = \delta^k_i \,, \quad \underset{(1)}{u}{}^k_\alpha = 0 \,, \quad \underset{(1)}{u}{}^k_a = X^\beta \left(\Gamma^k_{\beta a}\right)_p \,,$$

$$\underset{(\mu)}{u}{}^k_i =$$

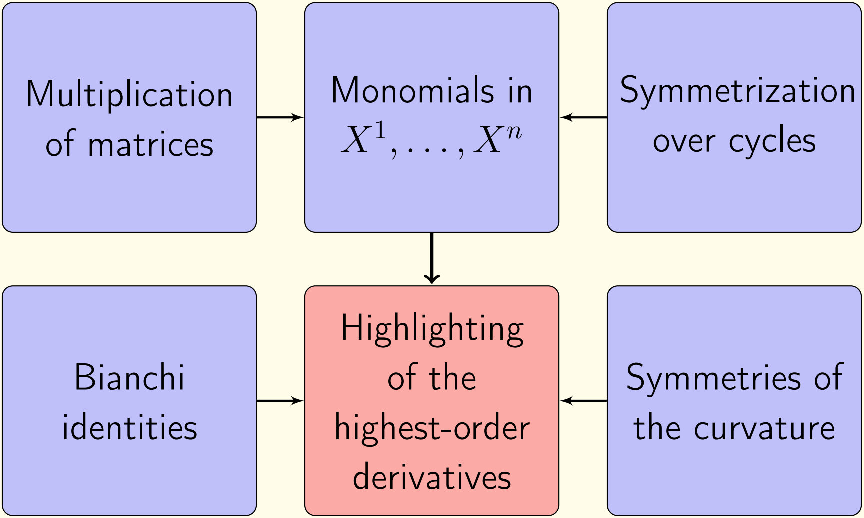
$$\frac{1}{\mu(\mu+\varepsilon(i))} \sum_{\sigma=2}^{\mu} \frac{1}{(\sigma-2)!} X^{\alpha_1} \ldots X^{\alpha_\sigma} \left(R^k_{\alpha_1\alpha_2 l\,;\,\alpha_3\ldots\alpha_\sigma}\right)_p \underset{(\mu-\sigma)}{u}{}^l_i \,,$$

$$\mu \geq 2 \,, \quad \varepsilon(\alpha) = 1 \,, \quad \varepsilon(a) = -1 \,,$$

Some problems in the general statement

- Spacetime metric is known. It is necessary to find the metric in some domain in normal coordinates (eg., in the tube neighborhood of the worldline of a particle).
- Spacetime metric has a high-dimensional isometry group (eg., in a spherically symmetric spacetime). It is necessary to solve the Cauchy problem with initial data on some spacelike or null hypersurface using the covariant expansions.
- It is necessary to find the formal covariant power series solution of the Einstein equations in a spacetime with some additional conditions (eg., in static or stationary spacetimes).

Outline of the algorithm



Time computational complexity

Let N be the degree of a monomial.
An arbitrary monomial has the form
$(X_1)^{A_1}\left(X^2\right)^{A_2}\ldots(X^n)\,A_n, \quad A_1+A_2+\ldots+A_n=N,$

The computational complexity of the algorithm is not worse than exponential, $O\left(2^N\right)$, in N. This estimation can be obtained by using the Hardy–Ramanujan formula for the number of partitions of N.

The computational complexity of the algorithm is factorial, $O(n!)$, in the dimension n of O_p.