# New Possibilities and Applications of the Method of Collocations and the Least Squares

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Sometimes it is useful to combine known methods to expand their capabilities. Such combinations are LSFEM (Least-Squares Finite Element Method) and CLS Method (Collocation and Least-Squares Method).
What can we get from a combination of the collocation method (CM) and least squares (LS) one?

**Problem:** Let several discrete values $u_i$, $i = 1, ..., m$ of smooth one-dimensional function $u(x)$ be given with an error. It is required to construct its effectively calculated approximant $u_a(x)$.

It is convenient to seek the solution of this problem in the form of a polynomial with indeterminate coefficients. Often in this case, interpolating polynomials (Newton, Lagrange, ...) give an error exceeding the error of the input data.
But if in this case we take in the input data the number of values of function $m$ more than the number of coefficients $n$ of the required polynomial $(m > n)$, then we can construct an approximant more accurate than the interpolant.

Gauss in this case reduced the construction of the approximant to the solution of the overdetermined SLAE

$$Ax = b \tag{1}$$

with a rectangular matrix of full rank with size $m \times n$, $(m > n)$.
In the general case, SLAE (1) does not have an exact solution, i.e. there is no such vector $y$ that $Ay \equiv b$. Therefore, discrepancy $Ay - b = r(y)$, $r(y) \neq 0$.
It is proposed for "solution" of task (1) to take solution of SLAE

$$A^T Ax = A^T b, \tag{2}$$

where $T$ denotes transposition of the matrix, $A^T A$ is a normal square matrix with size $n \times n$. The solution of problem (2) $x_{LS}$ is such that the residual functional $\Phi(r) \stackrel{def}{=} \sum_{i=1}^{n} r_i^2$ on it reaches its minimum in the linear space $\{x\}$.
In a sense, $x_{LS}$ is a more accurate solution of problem (1) than any other vector $y$.

Problem (1) has several ways of finding the solution $x_{LS}$. There is an orthogonal matrix $Q$ such that the first $n$ rows of $QA$ constitute an upper triangular matrix $R$ with size $n \times n$ and

$$Rx = Qb, \tag{3}$$

$x \equiv x_{LS}$. System (3) with triangular matrix $R$ has an easy solution. Какой способ из двух названных для решения линейной задачи наименьших квадратов (1) лучше? Which method of the two named for solving the linear least-squares problem (1) is better?

Consider the condition number of matrix $A$:

$$\nu(A) = \sqrt{\| A_1 \| \cdot \| A_1{}^{-1} \|}, \quad A_1 = A^T A. \tag{4}$$

It is known that $\nu(A) = \nu(R)$ and $\nu(A^T A) = \nu^2(A)$, i.e. SLAE (2) is worse conditioned than (3). When solving problems on a computer, the method of normal equations (2) can give an unacceptable error, unlike the $QR$ decomposition (3) method. **(Demmel J., Applied numerical linear algebra // Society for Industrial and Applied Mathematics, 1997. 430 p.)**

We use the $QR$ decomposition method in the CLS.

We consider the boundary value problem for PDE system

$$Lu(x) = f(x), \qquad x \in \Omega, \tag{5}$$

in domain $\Omega$ with boundary conditions

$$lu(x) = g(x), \qquad x \in \partial\Omega, \tag{6}$$

where $L$ and $l$ are differential operators,
$u = (u_1, u_2, ..., u_m)$ is the required vector-function,
$x = (x_1, x_2, ..., x_n)$, $x_1, x_2, ..., x_n$ are independent variables,
$f$ and $g$ are given vector-functions,
Usually nonlinear $L$, $l$ are linearized, and the numerical methods for solving (5) and (6) include iterations by nonlinearity.

Henceforth, we will call (5), (6) a differential problem.

Methods of collocation (CM), collocation and least squares CLS are grid, projection methods. In them, problem (5), (6) and its solutions are projected into a finite-dimensional linear functional space. Most often, a space of polynomials $\mathbb{P}$ is chosen. An approximant of the solution is sought as a combination with indefinite coefficients of the space basis elements $\phi_i(x)$:

$$u_a(x) = \sum_{i=0}^{n} c_i \phi_i(x). \tag{7}$$

Substitution of $u_a(x)$ in (5), (6) gives the equations to find it:

$$Lu_a = f(x), \quad x \in \Omega, \qquad lu_a(x) = g(x), \quad x \in \partial\Omega. \tag{8}$$

We will call problem (8) «approximate».

Its solution determines the approximate solution of the differential problem.

In CM и CLS,

*collocation equations* $(\boldsymbol{CE})$

$\boldsymbol{L_c \overset{def}{=} (Lu_a - f(x))|_{x=x_c} = 0}$, $\quad \boldsymbol{x_c \in \Omega}$,

$\boldsymbol{x_c = \{x_{c1}, x_{c2}, ..., x_{ck}\}}$ are *collocation points*,

*collocation boundary conditions* $(\boldsymbol{BCE})$ $\boldsymbol{l_b \overset{def}{=} (Lu_a - g(x))|_{x=x_b} = 0}$, $\boldsymbol{x_b \in \partial\Omega}$

$\boldsymbol{x_b}$ are *collocation points of boundary conditions*.

$$\begin{cases} \boldsymbol{L_c = 0}, \\ \boldsymbol{l_b = 0} \end{cases} \tag{9}$$

is a system of linear algebraic equations (SLAE) with respect to $\boldsymbol{c_i}$.

Substitution of SLAE (9) solution in (7) yields an approximant of problem (5), (6) solution.

In pseudospectral $\boldsymbol{p}$ version of the method, a single piece of the approximate solution is constructed in the entire region $\boldsymbol{\Omega}$. In grid $\boldsymbol{hp}$ and $\boldsymbol{h}$ versions, a grid is constructed in domain $\boldsymbol{\Omega}$. Domain *Omega* can also be partitioned into subdomains $\boldsymbol{\Omega_i},\ \boldsymbol{i = 1, ..., N}$. Each subdomain can contain one cell or a union of several cells. Henceforth, for brevity, subdomains will be called «cells».

A separate piece of the approximate global solution of problem (5), (6) is constructed in each cell $\boldsymbol{\Omega_i},\ \boldsymbol{i = 1, ..., N}$.

To find it, we write out an overdetermined SLAE, which we will call «local». The union of all local SLAEs will be called «global SLAE».

The conditionality of SLAE plays a key role in the ability to solve it by one method or another.

The conditionality of the global SLAE correlates with the conditionality of its local SLAEs: the worse the local SLAE is conditioned, the worse conditioned is the global SLAE, ....

In the CLS method on the common part of the border-sides of two adjacent cells, it is advisable to write down the matching conditions between the pieces of the solution in them.
In particular, they can take the form of continuity conditions for the solution, its various derivatives at individual points on the sides of two neighboring cells, belonging to both of them, etc.
It is advisable to take such matching conditions that after substitution in them the representation of the solution, (7) also become linear algebraic equations with respect to $\boldsymbol{c_i}$. It is advisable to include one part of these equations in the local SLAE of one of the neighboring cells, and the other part in the SLAE of the other cell.

The study of the conditionality of the SLAE obtained in the CLS method shows that the inclusion in it of matching conditions, which do not contradict the formulation of differential problem (5), (6), allows one to obtain SLAE significantly better conditioned than in the CM.

The global SLAE can be solved by direct or iterative methods, using the Schwartz alternating method for subdomains. In the latter case, local SLAEs are solved by direct methods in each cell. In one global iteration, all the cells of the computational domain are successively handled.

If values of the solution pieces found at the current global iteration are used in the matching conditions, then such a process is called Gauss-Seidel iterations, otherwise Gauss-Jacobi.

Matrix of the global SLAE is sparse. In the case of a grid with quadrangular cells, it is made up of five block diagonals. As shown by numerical experiments in the CLS method, it time-saving to solve the global SLAEs at $n \sim 100$ by direct methods.

If it is essential to take into account the sparseness of the matrix in the solution algorithm, then direct methods can solve the problem about ten times faster than iterations.

It is preferable to solve the resulting linear least squares problem by using the ***QR*** decomposition of the SLAE matrix (Givens, Householder) of the approximate problem. This approach has a significant advantage over the method of normal equations (Demmel J.) when solving insufficiently conditioned problems and their corresponding SLAEs and expands the capabilities of the CLS method for solving application tasks.

The technique of applying the CLS method is somewhat more complicated than using the CM. But, in view of the good conditionality of SLAE in the CLS method, it often allows solving problems with singularities and poorly conditioned ones, which can not be solved by other methods (CM, ...).

The presence of the analytical solution allows relatively simple construction of variants of the method of increased accuracy in various domains and on various grids, effectively using multigrid variants (etc.).

The requirement of minimizing the residual functional suppresses «non-physical» oscillations in generalized solutions of problems.

The CLS method can be conveniently and effectively parallelized without applying to the solution of approximate operations such as its interpolation and extrapolation between adjacent subdomains. ....

These and other properties of the method make it possible to effectively apply multigrid complexes, Krylov subspaces, preconditioners in the CLS method to accelerate the solution of problems. The combined application of these algorithms in the CLS method allowed the solution of the Navier-Stokes equations on the PC to be accelerated at Reynolds numbers 1000-2000 up to 50=300 times.

See more detail in **V.P. Shapeev and E.V. Vorozhtsov. Symbolic-numerical optimization and realization of the method of collocations and least residuals for solving the Navier-Stokes equations. Lecture Notes in Computer Science, Vol. 9890. Springer, Cham, 2016. - P. 473-488.**

The idea to combine CM with LS for solving PDE was proposed by Sleptsov A.G. (CM & LS => CLS).
It was implemented in work
**Plyasunova A. V. and Sleptsov A. G. Collocation Grid Method for Solving Nonlinear Parabolic Equations on Moving Grids. Model. Mekh, 1(18) (1987), N. 4, 116–137 (in Russian).**

(FEM & LS => LSFEM).
A large bibliography is given in book
**Bo-nan Jiang. The Least-Squares Finite Element Method. 1998. 418 p.**

About six hundred (600) publications are devoted to the LSFEM method.

Substitution of the found approximant into equations (5), (6) gives their *discrepancies*

$r_e(x) = Lu_a(x) - f(x), \quad r_b(x) = lu_a(x) - g(x).$

We consider a sequence of approximations $u_a^n(x)$, $n = 1, 2, \dots$ to the solution of problem (5), (6), which generates a sequence of discrepancies $r_e^n(x)$, $r_b^n(x)$.

In special cases of linear problems with certain restrictions on the left and right sides of equations (5), (6), at $\lim\limits_{n\to\infty} \|r_e^n(x)\| = 0$ and $\lim\limits_{n\to\infty} \|r_b^n(x)\| = 0$, it is proved that $\lim\limits_{n\to\infty} \|u(x) - u_a^n(x)\| = 0$ (**Jackson, Vainniko**).

In this case, there is an exponential convergence of polynomial approximants of solutions with increasing degrees $n$ of polynomials of the basis in $\mathbb{P}$.

That is, there are examples of the theoretical substantiation of these methods. **Vainniko G. M. On the stability and convergence of the collocation method. Differentsial'nye Uravneniya. 1, 244-254 (1965).**

In the works on application of numerical methods it is desirable to show the convergence of the approximate solution of the problem.

## Example of problem with large solution gradients

$$\begin{cases} \Delta v = -100(x_1^2 sin(10x_1x_2) + x_2^2 sin(10x_1x_2)), \\ \Omega = [0,1] \times [0,1], \quad v|_{\delta\Omega} = sin(10x_1x_2). \end{cases} \tag{10}$$

with known exact solution $u_{ex}(x_1, x_2) = \sin(10\, x_1 x_2)$ (Fig. 1).

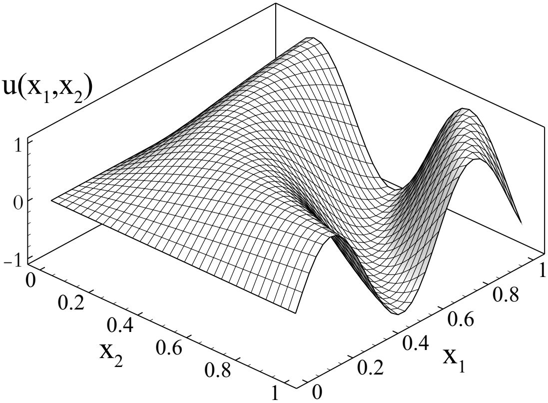



Рис. 1: The view of solution $u_{ex}$ of problem (10).

In p-version of the CLS, the error of the solution $\boldsymbol{E_c = 3.52 \cdot 10^{-15}}$ at $\boldsymbol{k = 26}$, the calculation time on the PC is 28 sec. In $\boldsymbol{h}$-version with the basis of polynomials of the second degree $\boldsymbol{E_c = 6.37 \cdot 10^{-5}}$ with the number of cells $\mathbf{30 \times 30}$, the calculation time is 272sec (on the same PC it is almost 10 times more!).

При больших значениях $\boldsymbol{k}$ For large values of $\boldsymbol{k}$
время решения задачи, достижимая точность, ошибки округлений зависят
the time of solving the problem, achievable accuracy, rounding errors depend on
1) the choice of the space of polynomials (products of monomials, products of orthogonal polynomials, ...),
2) arrangement of the points of collocations (uniformly, in the roots of orthogonal polynomials, ...)
3) the method of solving the SLAE,
4) the form of polynomials and the sequence of actions for their calculation.

We apply to the solution of the problem (10) $p$–version of the CLS method in the space with another basis:

$$u_a(x_1,x_2)=\sum_{i_1=0}^{N_1-1}\sum_{i_2=0}^{N_2-1} c_{i_1 i_2}\phi_{i_1}(x_1)\phi_{i_2}(x_2), \quad (x_1,x_2)\in[-1;1]\times[-1,1], \tag{11}$$

$\phi_{i_1}(y_1)=T_{i_1}(y_1), \quad \phi_{i_2}(y_2)=T_{i_2}(y_2)$ – Chebyshev polynomials of degree $i_1$, $i_2$, the collocation points in the roots of polynomials $T_{N_1}$, $T_{N_2}$. The results of high accuracy are in the following table.

Таблица 1: The error in numerical calculations by the $p$ – CLS method with a single cell for boundary value problem (??).

| $N_1 \times N_2$ | $Er(u)$ | $Er(u_x)$ | $Er(u_{xx})$ |
|---|---|---|---|
| 5 × 5 | 3.89e+0 | 6.53e+0 | 4.48e+0 |
| 10 × 10 | 5.95e–2 | 1.62e–1 | 4.06e–1 |
| 15 × 15 | 2.15e–5 | 8.65e–5 | 6.06e–4 |
| 20 × 20 | 2.04e–9 | 8.90e–9 | 4.55e–8 |
| 22 × 22 | 3.14e–11 | 1.82e–10 | 1.05e–9 |
| 24 × 24 | 4.87e–13 | 3.24e–12 | 2.28e–11 |
| 25 × 25 | 3.10e–14 | 1.52e–13 | 1.93e–12 |

A more difficult task (large gradients on the right side of the equation)

$$\frac{\partial^2 u(x_1,x_2)}{\partial x_1^2} + \frac{\partial^2 u(x_1,x_2)}{\partial x_2^2} = \frac{\partial^2 u_{ex}(x_1,x_2)}{\partial x_1^2} + \frac{\partial^2 u_{ex}(x_1,x_2)}{\partial x_2^2}, \quad (12)$$

$$u(x_1,x_2) = u_{ex}(x_1 x_2), \quad (x_1,x_2) \in \partial\Omega, \quad \Omega = [-1;1] \times [-1;1]$$

with exact solution ($u_{ex}(x_1,x_2)$ is the Runge function)

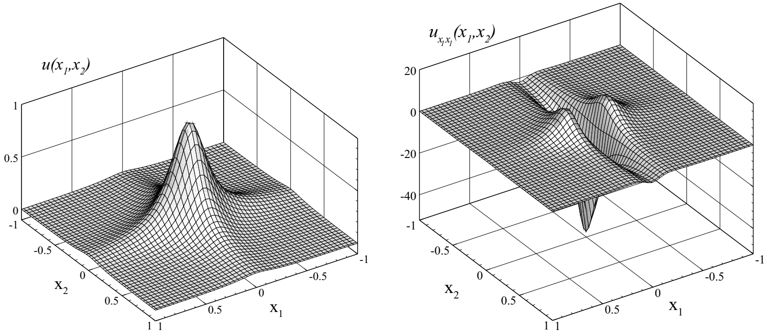
$$u_{ex}(x_1,x_2) = \frac{1}{(1+25\,x_1^2)}\frac{1}{(1+25\,x_2^2)}. \quad (13)$$

The right-hand side (12) grows rapidly in the neighborhood of zero.

(Cheney, W. A Course in Approximation Theory, / W. Cheney, W. Light Brooks/Cole. - 2000. - P. 360.) For $u_a(x_1,x_2) =$ as a product of two one-dimensional polynomials

$$u_a(x_1,x_2) = \sum_{i_1=0}^{N_1-1} \sum_{i_2=0}^{N_2-1} c_{i_1 i_2} \phi_{i_1}(x_1)\phi_{i_2}(x_2), \quad (x_1,x_2) \in [-1;1] \times [-1,1], \quad (14)$$

$\phi_{i_1}(y_1) = T_{i_1}(y_1), \quad \phi_{i_2}(y_2) = T_{i_2}(y_2)$ are Chebyshev polynomals of degree $i_1$, $i_2$, the collocation points in the roots of polynomials $T_{N_1}$, $T_{N_2}$.




a) b)

Рис. 2: a) the view of function $\boldsymbol{u_{ex}(x_1, x_2)}$ (13) and b) of its second derivative with respect to variable $\boldsymbol{x_1}$.

To solve problem (12), $p$–version of the CLS. The results are in table (2).

Таблица 2: P - CLS calculation error for problem (12).

| $N_1 \times N_2$ | $N$ | $Er(u)$ | $Er(u_x)$ | $Er(u_{xx})$ |
|---|---|---|---|---|
| 10 × 10 | 100 | 2.67e+0 | 1.27e+0 | 4.61e+0 |
| 20 × 20 | 400 | 3.47e–1 | 3.88e–1 | 2.08e+0 |
| 40 × 40 | 1600 | 3.77e–3 | 1.51e–1 | 1.59e+0 |
| 80 × 80 | 6400 | 5.97e–6 | 5.17e–4 | 6.42e–3 |
| 100 × 100 | 10000 | 1.96e–7 | 3.79e–5 | 3.85e–4 |

Table 2 shows that the error in the numerical solution decreases more slowly than in the solution of the previous problem (Table 1). The degree of polynomials in the basis reached the 200th power. And up to this value, the solution consistently converged (!), thanks to a combination of a good basis and good collocation points **(Boyd, J.P. Chebyshev and Fourier Spectral Methods: Second Revised Edition. / J.P. Boyd Dover Publications.- 2001. - - P. 668.)** The further increase in the degree of polynomials does not improve accuracy due to the accumulation of rounding errors. To find a more accurate solution, we apply the ***hp***–version of the CLS with four cells.

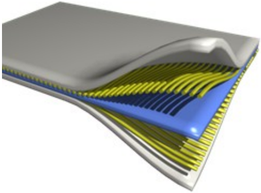
Таблица 3: The error of calculations using the $\boldsymbol{hp}-$CLS method with 4 cells solving problem (12).

| $N_1 \times N_2$ | $N$ | $Er(u)$ | $Er(u_x)$ | $Er(u_{xx})$ |
|---|---|---|---|---|
| 10 × 10 | 400 | 1.78e–1 | 9.89e–1 | 5.14e+0 |
| 20 × 20 | 1600 | 2.32e–4 | 2.97e–3 | 3.12e–2 |
| 40 × 40 | 6400 | 1.54e–9 | 7.92e–8 | 3.82e–6 |
| 50 × 50 | 10000 | 1.72e–12 | 8.24e–11 | 5.32e–9 |
| 60 × 60 | 14400 | 9.15e–14 | 2.82e–14 | 1.99e–10 |

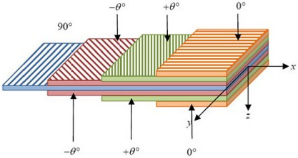
It can be seen from the table that the solution was obtained with a very high accuracy of the order of $\boldsymbol{Er(u) = 10^{-14}}$. (So simple: just 4 cells! Here, a comparison with published numerical solutions by other methods shows the advantage of the CLS method.

**Laminated anisotropic** plates are elements of many high-tech structures.

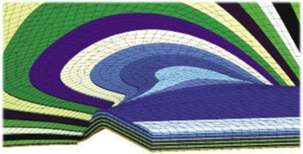
They are composed of layers of constant or variable thickness and material of each layer may have its own unique properties, including **anisotropy**.



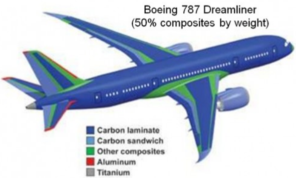
Different materials of layers




Controlled layers orientation scheme



Anisotropy


Boeing 787 Dreamliner
(50% composites by weight)
Carbon laminate
Carbon sandwich
Other composites
Aluminum
Titanium


A380
AIRBUS

# 2. 3D problem formulation

Let us consider a static bending of laminate composed of $\boldsymbol{N}$ layers of constant thickness. Layers are transversely isotropic with material symmetry axe in the plate's plane, causing anisotropy in plate. The upper surface of the plate is under uniform transverse load, the lower surface is free, and a continuity condition of displacements $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and stresses $\boldsymbol{\sigma_{zz}}$, $\boldsymbol{\sigma_{xz}}$, $\boldsymbol{\sigma_{yz}}$ is used on interface surfaces. The corresponding boundary conditions are defined on the boundary of the plate. The task is to calculate the stress and displacement fields of such plates.

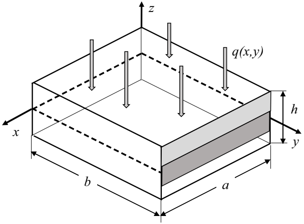
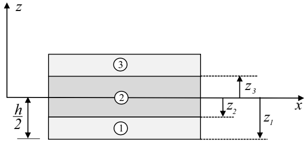
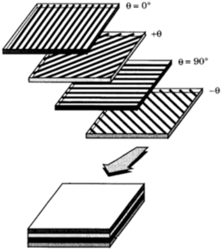



Plate under transverse loading




Layers of constant thickness




Layers orientation scheme

# 2. 3D elasticity governing equations

Each layer is defined by the elliptic system of 3 differential equations of 6 order in 3 variables.

## 3D elasticity governing equations in displacements for $k$-th layer

$$
\begin{aligned}
&Q_{11}^k \frac{\partial^2 u^k}{\partial x^2} + 2Q_{16}^k \frac{\partial^2 u^k}{\partial x \partial y} + Q_{11}^k \frac{\partial^2 u^k}{\partial y^2} + Q_{55}^k \frac{\partial^2 u^k}{\partial z^2} + (Q_{12}^k + Q_{66}^k) \frac{\partial^2 v^k}{\partial x \partial y} + \\
&Q_{16}^k \frac{\partial^2 v^k}{\partial x^2} + Q_{16}^k \frac{\partial^2 v^k}{\partial y^2} + Q_{45}^k \frac{\partial^2 v^k}{\partial z^2} + (Q_{13}^k + Q_{55}^k) \frac{\partial^2 w^k}{\partial x \partial z} + (Q_{36}^k + Q_{45}^k) \frac{\partial^2 w^k}{\partial y \partial z} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&Q_{16}^k \frac{\partial^2 u^k}{\partial x^2} + (Q_{12}^k + Q_{66}^k) \frac{\partial^2 u^k}{\partial x \partial y} + Q_{16}^k \frac{\partial^2 u^k}{\partial y^2} + Q_{45}^k \frac{\partial^2 u^k}{\partial z^2} + Q_{66}^k \frac{\partial^2 v^k}{\partial x^2} + \\
&+ 2Q_{26}^k \frac{\partial^2 v^k}{\partial x \partial y} + Q_{66}^k \frac{\partial^2 v^k}{\partial y^2} + Q_{44}^k \frac{\partial^2 v^k}{\partial z^2} + (Q_{36}^k + Q_{45}^k) \frac{\partial^2 w^k}{\partial x \partial z} + (Q_{23}^k + Q_{44}^k) \frac{\partial^2 w^k}{\partial y \partial z} = 0,
\end{aligned}
$$

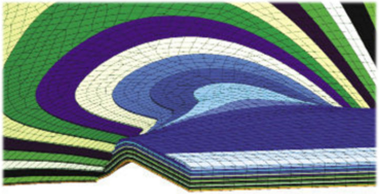
$$
\begin{aligned}
&(Q_{13}^k + Q_{55}^k) \frac{\partial^2 u^k}{\partial x \partial z} + (Q_{36}^k + Q_{45}^k) \frac{\partial^2 u^k}{\partial y \partial z} + (Q_{36}^k + Q_{45}^k) \frac{\partial^2 v^k}{\partial x \partial z} + \\
&+ (Q_{23}^k + Q_{44}^k) \frac{\partial^2 v^k}{\partial y \partial z} + Q_{55}^k \frac{\partial^2 w^k}{\partial x^2} + 2Q_{45}^k \frac{\partial^2 w^k}{\partial x \partial y} + Q_{55}^k \frac{\partial^2 w^k}{\partial y^2} + Q_{33}^k \frac{\partial^2 w^k}{\partial z^2} = 0.
\end{aligned}
$$

Here $Q_{ij}^k$ – stiffness coefficients of $k$-th layer.

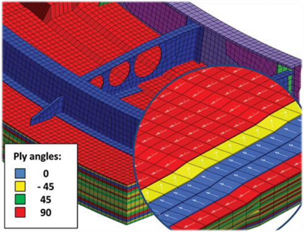
# 2. Calculation complexity of laminated structures

In the numerical solution of such problems a number of difficulties arise

- presence of **small parameters** in the derivatives due to
  - the smallness of the relative thicknesses of the layers ($\varepsilon_{\boldsymbol{k}}$)
  - the materials anisotropy ($\varepsilon_{\boldsymbol{m}}$)
- **dependence of the order of DE systems** on the number of layers.
- **fine spatial grid** to represent each layer.



Modeling blades of a jet engine from a laminated composite




Modeling fuselage element

# 3. 2D problem formulation. Plates theories

Small relative thickness allows to formulate some hypotheses on the distribution of displacements and stresses through the plate thickness, and thus reduce the 3D problem of elasticity theory to a 2D problem of plates theories.

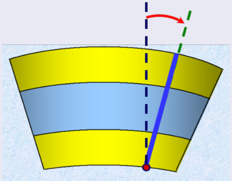
1. We select the reference plane in the plate.
2. Using the hypothesis, the problem reduces to the determination of displacement and stress fields of reference plane
3. Using the same hypothesis, we define displacement and stress fields of the entire plate

Solutions obtained in the framework of the theory of plates, are **approximations** to the solution of the 3D elasticity. The quality of the approximation is determined by theory choice.

# 3. Plates theories

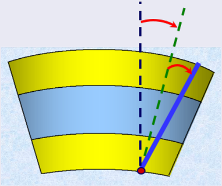
We consider **three** different theories of plates.

### Kirchhoff–Love theory



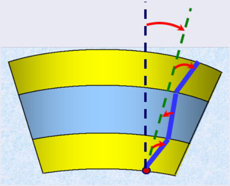
Direct normal hypothesis

### Timoshenko's theory



Direct line hypothesis

### Grigolyuk-Chulkov's theory



Broken line hypothesis

The normal to the middle surface elements **keeps**

**For entire plate**
- straightness
- length
- perpendicularity

**For separate layer**
- straightness
- length

**For layer plate**
- straightness
- length

In these theories transverse shear is simulates differently.

# 3. Governing equations of Kirchhoff–Love theory

The original equations are presented in the works of S.G. Lekhnitskiy, V.V. Novozhilov.

## These governing equations in displacements

System of 3 DE in **2 variables** of 8 order.

$$
\begin{aligned}
&(A_{12}+A_{66})\frac{\partial^2 v}{\partial x\,\partial y}+2A_{16}\frac{\partial^2 u}{\partial x\,\partial y}+(-B_{12}-2B_{66})\frac{\partial^3 w}{\partial x\,\partial y^2}-3B_{16}\frac{\partial^3 w}{\partial x^2\,\partial y}+A_{11}\frac{\partial^2 u}{\partial x^2}+\\
&+A_{16}\frac{\partial^2 v}{\partial x^2}+A_{26}\frac{\partial^2 v}{\partial y^2}+A_{66}\frac{\partial^2 u}{\partial y^2}-B_{11}\frac{\partial^3 w}{\partial x^3}-B_{26}\frac{\partial^3 w}{\partial y^3}=0,
\end{aligned}
$$

$$
\begin{aligned}
&A_{12}\frac{\partial^2 u}{\partial x\,\partial y}+2A_{26}\frac{\partial^2 v}{\partial x\,\partial y}+A_{66}\frac{\partial^2 u}{\partial x\,\partial y}+(-B_{12}-2B_{66})\frac{\partial^3 w}{\partial x^2\,\partial y}-3B_{26}\frac{\partial^3 w}{\partial x\,\partial y^2}+\\
&+A_{16}\frac{\partial^2 u}{\partial x^2}+A_{22}\frac{\partial^2 v}{\partial y^2}+A_{26}\frac{\partial^2 u}{\partial y^2}+A_{66}\frac{\partial^2 v}{\partial x^2}-B_{16}\frac{\partial^3 w}{\partial x^3}-B_{22}\frac{\partial^3 w}{\partial y^3}=0,
\end{aligned}
$$

$$
\begin{aligned}
&(B_{12}+2B_{66})\frac{\partial^3 v}{\partial x^2\,\partial y}+B_{12}\frac{\partial^3 u}{\partial x\,\partial y^2}+3B_{16}\frac{\partial^3 u}{\partial x^2\,\partial y}+3B_{26}\frac{\partial^3 v}{\partial x\,\partial y^2}+2B_{66}\frac{\partial^3 u}{\partial x\,\partial y^2}+\\
&(-2D_{12}-4D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2}-4D_{16}\frac{\partial^4 w}{\partial x^3\,\partial y}-4D_{26}\frac{\partial^4 w}{\partial x\,\partial y^3}+B_{11}\frac{\partial^3 u}{\partial x^3}+B_{16}\frac{\partial^3 v}{\partial x^3}+B_{22}\frac{\partial^3 v}{\partial y^3}+\\
&+B_{26}\frac{\partial^3 u}{\partial y^3}-D_{11}\frac{\partial^4 w}{\partial x^4}-D_{22}\frac{\partial^4 w}{\partial y^4}=-q(x,y).
\end{aligned}
$$

# 3. Special equations of Kirchhoff-Love plates theory

In special cases the general equations of Kirchhoff-Love plates theory take a simpler form.

- **Beam bending**

$$\frac{\partial^4 w(x)}{\partial x^4} = \frac{q(x,y)}{D}, \quad D = Ebh^3/12$$

- **Bending of rectangular isotropic plate**

$$\frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4} = \frac{q(x,y)}{D}, \quad D = Eh^3/12(1-\nu^2)$$

- **Bending of rectangular orthotropic plate**

$$D_{11}\frac{\partial^4 w(x,y)}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w(x,y)}{\partial y^4} = q(x,y)$$

# 3. Governing equations of Timoshenko's theory

The original equations are presented in the works of S.P. Timoshenko, A.T. Vasilenko, Y.M. Grigorenko.

## These governing equations in displacements

System of 5 DE in **2 variables** of 10 order.

$$(A_{12}+A_{66})\frac{\partial^2 v}{\partial x\,\partial y}+2A_{16}\frac{\partial^2 u}{\partial x\,\partial y}+(B_{12}+B_{66})\frac{\partial^2 \phi_y}{\partial x\,\partial y}+2B_{16}\frac{\partial^2 \phi_x}{\partial x\,\partial y}+A_{11}\frac{\partial^2 u}{\partial x^2}+A_{16}\frac{\partial^2 v}{\partial x^2}+$$
$$+A_{26}\frac{\partial^2 v}{\partial y^2}+A_{66}\frac{\partial^2 u}{\partial y^2}+B_{11}\frac{\partial^2 \phi_x}{\partial x^2}+B_{16}\frac{\partial^2 \phi_y}{\partial x^2}+B_{26}\frac{\partial^2 \phi_y}{\partial y^2}+B_{66}\frac{\partial^2 \phi_x}{\partial y^2}=0,$$

$$(A_{12}+A_{66})\frac{\partial^2 u}{\partial x\,\partial y}+2A_{26}\frac{\partial^2 v}{\partial x\,\partial y}+(B_{12}+B_{66})\frac{\partial^2 \phi_x}{\partial x\,\partial y}+2B_{26}\frac{\partial^2 \phi_y}{\partial x\,\partial y}+A_{16}\frac{\partial^2 u}{\partial x^2}+A_{22}\frac{\partial^2 v}{\partial y^2}+$$
$$+A_{26}\frac{\partial^2 u}{\partial y^2}+A_{66}\frac{\partial^2 v}{\partial x^2}+B_{16}\frac{\partial^2 \phi_x}{\partial x^2}+B_{22}\frac{\partial^2 \phi_y}{\partial y^2}+B_{26}\frac{\partial^2 \phi_x}{\partial y^2}+B_{66}\frac{\partial^2 \phi_y}{\partial x^2}=0,$$

$$-2A_{45}\frac{\partial^2 w}{\partial x\,\partial y}-A_{44}\frac{\partial \phi_y}{\partial y}-A_{44}\frac{\partial^2 w}{\partial y^2}-A_{45}\frac{\partial \phi_x}{\partial y}-A_{45}\frac{\partial \phi_y}{\partial x}-A_{55}\frac{\partial \phi_x}{\partial x}-A_{55}\frac{\partial^2 w}{\partial x^2}=-q,$$

$$(B_{12}+B_{66})\frac{\partial^2 v}{\partial x\,\partial y}+2B_{16}\frac{\partial^2 u}{\partial x\,\partial y}+(D_{12}+D_{66})\frac{\partial^2 \phi_y}{\partial x\,\partial y}+2D_{16}\frac{\partial^2 \phi_x}{\partial x\,\partial y}-A_{45}\phi_y-A_{45}\frac{\partial w}{\partial y}-A_{55}\phi_x-A_{55}\frac{\partial w}{\partial x}+$$
$$+B_{11}\frac{\partial^2 u}{\partial x^2}+B_{16}\frac{\partial^2 v}{\partial x^2}+B_{26}\frac{\partial^2 v}{\partial y^2}+B_{66}\frac{\partial^2 u}{\partial y^2}+D_{11}\frac{\partial^2 \phi_x}{\partial x^2}+D_{16}\frac{\partial^2 \phi_y}{\partial x^2}+D_{26}\frac{\partial^2 \phi_y}{\partial y^2}+D_{66}\frac{\partial^2 \phi_x}{\partial y^2}=0,$$

...

# 3. Governing equations of Grigolyuk-Chulkov's theory

The original equations are presented in the works of E.I. Grigolyuk[1,2].

## These governing equations in displacements

For $N$ layers system of DE in **2 variables** has $6 + 4N$ order.
Для однойслойных пластин разрешающие уравнения теории ломаной линии совпадают с уравнениями теории Тимошенко. For $N = 1$ governing equations of Grigolyuk-Chulkov's theory coincide with the equations of the theory of Timoshenko.

[1] Grigolyuk E.I., Chulkov P.L. Non-linear equations of thin elastic laminated anisotropic shallow shells with rigid filler // Izv. AS USSR: Mechanics. 1965. № 5. P.65–80.

[2] Grigolyuk E.I., Kulikov G.M. Multilayer reinforced shell. M.: Mashinostroenie, 1988.

# 3. Comparison of theories

| Theory | | DE order | Small parameters |
|---|---|---|---|
| Kirchhoff-Love theory | 2D | $\mathbf{8}$ | $O(1)$ |
| Timoshenko's theory | 2D | $\mathbf{10}$ | $O(\varepsilon^2)$ |
| Grigolyuk-Chulkov's theory | 2D | $6 + 4N$ | $O(\varepsilon_k^2\, \varepsilon_m)$ |
| 3D elasticity | 3D | $6N$ | $O(\varepsilon_k^2\, \varepsilon_m)$ |

Here:

- $N$ – number of layers; $h = \sum h^k$.
- $\varepsilon = h/a$ – ratio of plate thickness to the characteristic size in plane. In the real structures has the order of $10^{-2}$ to $10^{-3}$.
- $\varepsilon_k = h^k/a$ – ratio of layer thickness to the characteristic size in plane.
- $\varepsilon_m$ – the ratio of the elastic moduli in the main direction of the isotropy and perpendicular. The carbon plastics is of the order $10^{-2}$.

# The idea of the collocation and least squares method

Collocation method + Least squares method = Collocation and least squares method

## $P$–approach

The use of polynomials of high degrees in the CLR method.

Collocation and least squares method (CLR) + $p-$approach = $hp$-version of CLR method

# 4. Collocation and least squares method

**Collocation and least squares method (CLR)** – collocation method, which is considered a more general approach to minimizing the residual functional.

In the standard implementation of the method minimization of the functional in the sense of least squares is used

$$\sum_{i=0}^{N^*}\left[Lu_a(x_{col}^i)-f(x_{col}^i)\right]^2 \to min$$

где $L$ – differential operator, $u_a$ — approximate solution, $f$ – known right-hand sides, $x_{col}$ – collocation nodes.

SLAE in CLR method is overdetermined due to larger number of collocation points.

# 4. Description of the CLR method

Consider the canonical area (line, rectangle, parallelepiped)

- The considered area covered by the grid.
- In tach grid cell we define a local coordinate system such that local variables vary from -1 to 1.
- In the cells of the grid approximate solution is a linear combination of basis functions

$$(u_a)_k(y) = \sum_{j=0}^{N^k} c_{kj}\, \varphi_j(y), \qquad k = 1, \ldots, K,$$

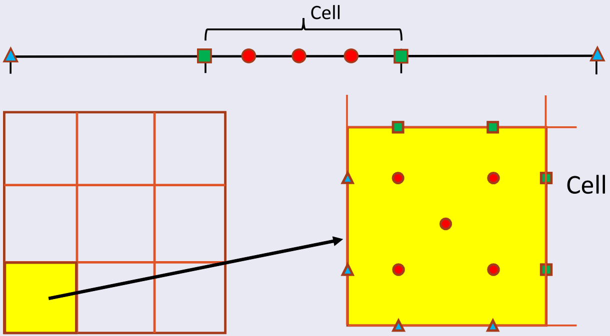
where $K$–cell number, $N^k$ – the number of basis functions in $k$-th cell.

# 4. Building SLAE

To find the unknown coefficients in each cell is we build SLAE of the following equations:

- Collocation equtions
- Boundary equations
- Matching conditions

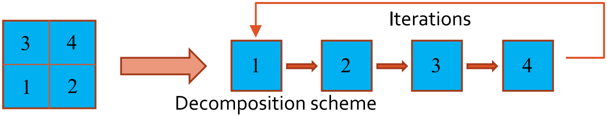
## The alignment of points in 1D and 2D cases



# 4. Solution of SLAE

SLAE in cell is **overdetermined**.

- If it possible to solve the overdetermined linear systems in the least squares sense we use $\boldsymbol{QR}$ factorization of its matrix, implemented by Householder method.
- Otherwise, use **decomposition method** - alternating method of Schwarz. It allows to reduce the problem in the entire region to the task sequence in subdomains. As a result, a separate problem in the subregion has SLAE of **small size** and can be solved by the direct method.



# 5. Strategies for reducing the error

## Estimation of approximation error in the collocation method

For boundary value problem of $m$th order with sufficiently smooth solution the approximation error estimates

$$||u_a - u||_\infty = O(h^{p-m+1})$$

Thus **three** strategies to reduce the approximation error follows

- $h \to min$, piecewise polynomial approximation (***h*–approach**).
- $p \to max$, increasing the degree of the polynomial (***p*–approach**).
- simultaneous variation $h$ и $p$ (***hp*–approach**).

# 5. Strategies for reducing the error

### $h$–approach

- When reducing the $h$ and $p$ fixed the error is reduce algebraically.
- Resolution SLAE is large, but it is sparse.

### $p$–approach

- When increasing polynomials degree $p$ the error reduce exponentially.
- Resolution SLAE is smaller, but it is completely filled.

### $hp$–approach

Allows to obtain the necessary accuracy at a lower computational cost with compared with $h$ - and $p$ - approaches.

# 6. Selection of collocation points in ***hp***–version of CLR method

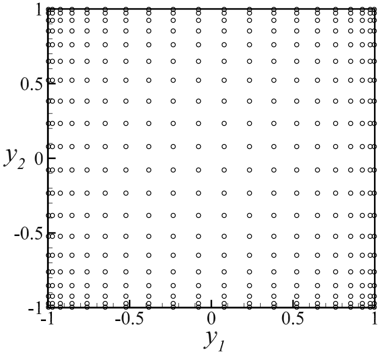
In ***hp***–version of CLR method local coordinates of the collocation points are selected as the $\alpha_i$ -**roots** of Chebyshev polynomial of degree $N$.

**1D case**

$$(\alpha^i),\ i = 1, .., N$$

**2D case**

$$(\alpha^i, \alpha^j),$$
$$i, j = 1, .., N$$




Arrangement of collocation points in the two-dimensional case ($N$=20)

# 6. Selection of the basis of presentation in *hp*–version of CLR method

In the 1D case, the solution is represented as a linear combination of

$$u_a(y_1) = \sum_{i=0}^{N_1-1} c_i T_i(y_1),$$

where $T_n$ – Chebyshev polynomials of the first kind of degree $n$.

In the 2D case in a square area of the solution it is represented as a direct product two 1D representations

$$u_a(y_1, y_2) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} c_{ij} T_i(y_1) T_j(y_2).$$

An approximate solution is presented in the form of a direct product.

# 7. Verification. The test problem.

To verify the method, we consider the case of a quadratic simply supported plate. In this case, we know the exact solution in the framework of the Kirchhoff-Love theory[1].

Таблица 4: The value of deflection at the center of the plate, obtained by $hp-$ version of CLR method.

| $N_1 \times N_2$ | $w_0(0.5, 0.5)/q_0 \cdot 10^8$ |
|---|---|
| Exact | 9.857712 |
| **10 × 10** | **9.857**9 |
| **20 × 20** | **9.8577**3 |
| **30 × 30** | **9.85771**5 |

[1] Timoshenko S. and Woinowsky-Krieger S. Theory of plates and shells. McGraw–Hill New York, 1959.

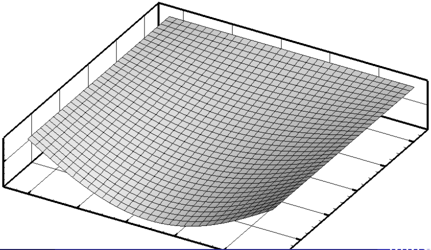
# 7. The plate with the free edge

It is also considered the case when one of the edges ($x = 1$) is free from restraints. In this case, the boundary conditions take the form of a linear combination of derivatives of higher order:

$$D_{11}\frac{\partial^2 w_0}{\partial x^2} + D_{12}\frac{\partial^2 w_0}{\partial y^2} = 0,$$

$$D_{11}\frac{\partial^3 w_0}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w_0}{\partial x^2 \partial y} = 0.$$

These boundary conditions are easily implemented in the *hp*−version of CLR method.



# 7. Calculation of laminated plates

We consider 3-layer plate with lamination scheme 0/90/0. Parameters of material

$$E_L = 25E_T, \quad G_{LT} = 0.5E_T, \quad G_{TT} = 0.2E_T, \quad \nu_{LT} = \nu_{TT} = 0.25,$$

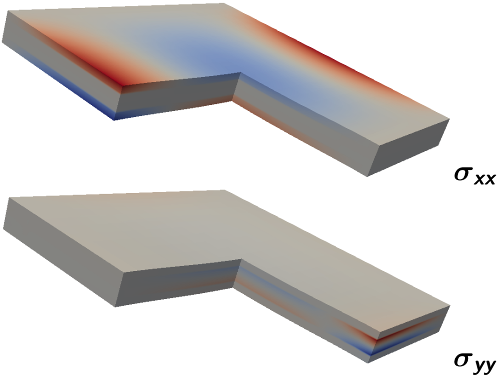
where $L$, $T$– the direction along and across the symmetry axe, respectively.

## Boundary conditions

- On the side faces plate is clamped.
- At the boundary between the layers we use the condition of ideal contact.
- The lower bound is free.
- On the upper bound uniformly distributed load $q_0$ is applied.

# 7. Stresses in 3-layer clamped plate

The main contribution to the stress fields make by $\sigma_{xx}$ и $\sigma_{yy}$. The greatest stresses occur at the edges of the plate, which is set to fix the condition.




Stress fields in 3-layer clamped plate

# 7. Stresses and displacements in 3-layer clamped plate

Stresses and deflection in 3-ply laminate. The results of calculations carried out in the framework of plate theories: Kirchhoff-Love (KLT), Timoshenko's (TT) and Grigolyuk-Chulkov's (GCT). Sign (%) is used for relative percentage deviation from GCT.

| $h/a$ | GCT | TT | KLT | TT (%) | KLT (%) |
|---|---|---|---|---|---|
| | | | $\bar{w}(a/2, a/2, h/2)$ | | |
| 0.1 | –0.751 | –0.603 | –0.208 | 31.30 | 38.36 |
| 0.05 | –2.650 | –2.538 | –1.667 | 2.28 | 4.20 |
| 0.02 | –28.523 | –28.335 | –26.044 | **0.04** | 1.74 |
| 0.01 | –213.312 | –212.974 | –208.354 | **0.02** | **0.62** |
| | | | $\bar{\sigma}^3_{xx}(a, 0, h/2)$ | | |
| 0.1 | 0.975 | 0.475 | 0.579 | 51.2 | 40.6 |
| 0.05 | 0.569 | 0.545 | 0.579 | 4.28 | 1.61 |
| 0.02 | 0.576 | 0.574 | 0.578 | **0.30** | **0.42** |
| 0.01 | 0.578 | 0.578 | 0.578 | **0.05** | **0.13** |
| | | | $\bar{\sigma}^2_{yy}(0, a, h/4)$ | | |
| 0.1 | 0.636 | 0.662 | 0.456 | 4.21 | 28.2 |
| 0.05 | 0.605 | 0.536 | 0.456 | 11.5 | 24.6 |
| 0.02 | 0.483 | 0.471 | 0.456 | 2.65 | 5.72 |
| 0.01 | 0.463 | 0.460 | 0.455 | **0.64** | **1.82** |

# Comparison of plate theories and elastic theory calculations

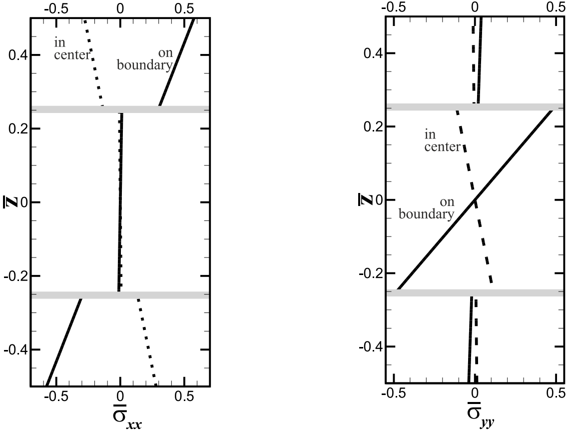
The calculations we used one cell with 16x16 collocation points.
The table shows the values of the deflection and stressess that was calculated in the framework of frequently used plate theories: Kirchhoff-Love (KLT), Timoshenko's (TT) and Grigolyuk-Chulkov's (GCT). Here (%) is the difference between the theory of plates and the elasticity theory in percentages.

| $h/a$ | 3D Elastic (1) | KLT | TT | GCT | KLT (%) | TT (%) | GCT (%) |
|---|---|---|---|---|---|---|---|
| | | | $w(a/2, b/2, 0)$ | | | | |
| 0.25 | 2.618(-5) | 4.680(-6) | 2.200(-5) | 2.544(-5) | **82.12** | 15.95 | **2.81** |
| 0.1 | 1.333(-4) | 7.310(-5) | 1.139(-4) | 1.306(-4) | 45.16 | 14.55 | **2.02** |
| 0.05 | 7.078(-4) | 5.850(-4) | 6.710(-4) | 7.015(-4) | 17.35 | 5.20 | **0.89** |
| 0.02 | 9.427(-3) | 9.140(-3) | 9.230(-3) | 9.421(-3) | 3.05 | 2.09 | **0.07** |
| 0.01 | 7.368(-2) | 7.310(-2) | 7.341(-2) | 7.360(-2) | **0.79** | **0.37** | **0.11** |
| | | | $\sigma_{yy}(0, 0, -h/4)$ | | | | |
| 0.25 | 1.744(5) | 4.032(4) | 1.494(5) | 1.712(5) | 76.88 | 14.31 | **1.83** |
| 0.1 | 4.180(5) | 2.520(5) | 3.740(5) | 4.100(5) | 39.71 | 10.53 | **1.91** |
| 0.05 | 1.176(6) | 1.008(6) | 1.128(6) | 1.168(6) | 14.29 | 4.08 | **0.68** |
| 0.02 | 6.475(6) | 6.300(6) | 6.375(6) | 6.425(6) | 2.70 | 1.54 | **0.77** |
| 0.01 | 2.530(7) | 2.520(7) | 2.520(7) | 2.530(7) | **0.40** | **0.40** | **$\leq$0.01** |

**(1) Pagano N. J.** Exact Solution for Rectangular Bidirectional Composites and Sandwich Plates // J. Composite Materials. 1970. Vol 4. P. 20-34.

# 7. Stresses and displacements in 3-layer clamped plate

Stresses distribution along $z$ coordinate $\bar{\sigma}_{xx}(a,0,\bar{z})$, $\bar{\sigma}_{xx}(a/2,a/2,\bar{z})$ and $\bar{\sigma}_{yy}(0,b,\bar{z})$, $\bar{\sigma}_{yy}(a/2,a/2,\bar{z})$ in 3-ply laminate for $h/a = 0.02$.




The maximum absolute values of the stresses at the edges of the plate are observed.

# Заключение

1. A *hp* - version of the collocations and least squares method based on approximation by polynomials of high degrees is developed. As a collocation points roots of Chebyshev polynomials are used, and the basis is represented as a direct product of the series of Chebyshev polynomials.
2. The proposed numerical method is implemented for the calculation of bending problem of laminated anisotropic rectangular plates.
3. The problem of bending of clamped rectangular laminated plates with transversely isotropic layers under a uniform load is solved. The calculation was performed in the framework of the Kirchhoff-Love and Tymoshenko's and refined Grigolyuk-Chulkov's plate theories. The analysis of the results was carried out.

Thank you for attention!

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# High-precision solutions to the problem of flow in a cavity

High-precision solutions to the problem of flow in a cavity, obtained in the following papers.

**(1)** Botella O. and Peyret R. (1998) (Spectral method)

To increase the accuracy, they use the elimination of the principal terms of the solution asymptotics in the upper corners of the cavity, where the velocity suffers a discontinuity).

**(2)** Garanzha V.A., Kon'shin V.N. (1999)

(Finite-difference scheme of the fourth order of approximation.)

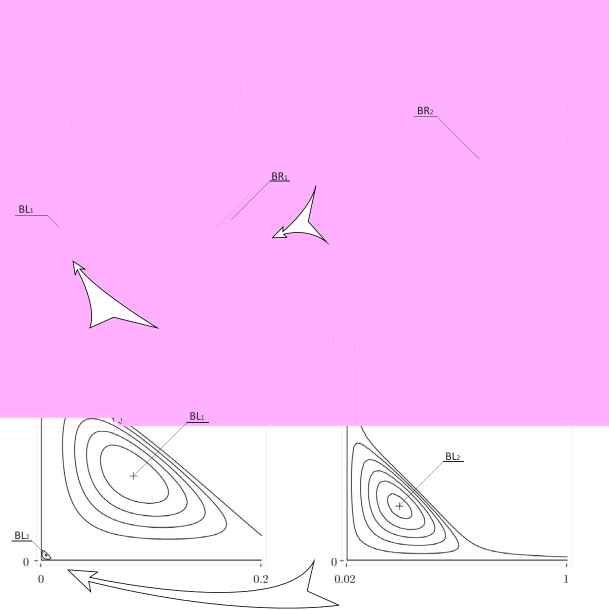
**(3)** Shapeev A.V., Lin P. (2009)

(High-precision FEM)

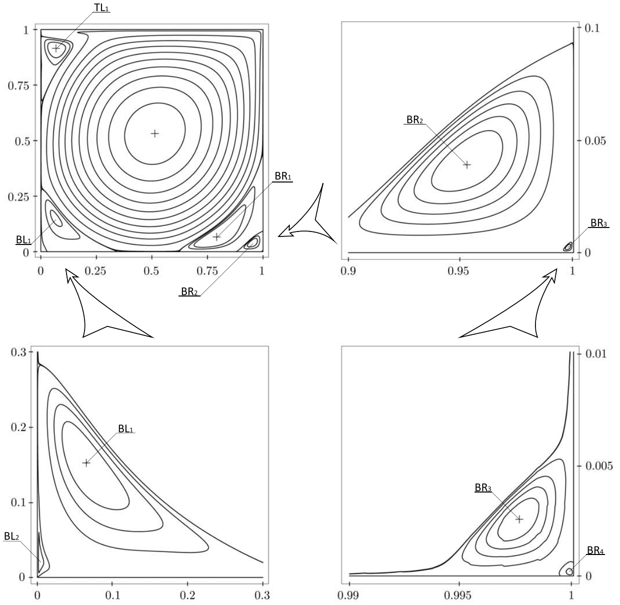
Таблица 5: Fragment of comparison of calculations. The ***PE*** vortex characteristics obtained by the CLS method* on a sequence of refining grids and by other authors ($\boldsymbol{Re} = 1000$).

| работа | $\psi$ | $x_1$ | $x_2$ |
|---|---|---|---|
| (*) сетка $M_1$ | **−0.11885323** | **0.53067831** | **0.56523414** |
| (*) сетка $M_2$ | **−0.11893562** | **0.53078734** | **0.56523714** |
| (*) сетка $M_3$ | **−0.11893658** | **0.53079011** | **0.56524057** |
| **(1)** | **−0.1189366** | **0.5308** | **0.5652** |
| **(2)** | **−0.118938** | **0.5300** | **0.5650** |
| **(3)** | **−0.1189366** | **0.5307901** | **0.5652406** |

The numbers in the third and sixth rows of the table coincide with an accuracy of $\sim \mathbf{10^{-8}}$.




**Flow pattern in the cavity at $\mathrm{Re} = 1000$ and its enlarged fragments. Received by the CLR method ($m_v = 2$, $m_p = 2$) on the Gaussian-Lobatto grid $320 \times 320$.** (Vortex names are given in **Ghia U., Ghia K.N., Shin C.T.** High-Re Solutions for Incompressible ... JCP, 1982.)




Flow pattern in the cavity at $\mathrm{Re} = 7500$ and its enlarged fragments.

The approximate solution is obtained by the CLS method ($m_v = 2$, $m_p = 1$) on uniform grid $1280 \times 1280$.