Modeling Quantum Behavior in the Framework of Permutation Groups

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Vladimir Kornyak

Laboratory of Information Technologies
Joint Institute for Nuclear Research
Dubna, Russia

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1. Quantum mechanics

2. Constructive modification

3. Modeling quantum evolution

Quantum mechanics I. States

1. Pure state = ray in Hilbert space $\mathcal{H}$ over $\mathbb{C}$

   equivalence $\sim$: $|\psi\rangle \sim a\,|\psi\rangle$ $\qquad |\psi\rangle \in \mathcal{H},\ a \in \mathbb{C}$

   $\xrightarrow{\text{reducing } \sim \text{ by normalization}} |\psi\rangle \sim e^{i\alpha}\,|\psi\rangle$ $\qquad \|\psi\| = 1,\ \alpha \in \mathbb{R}$

   $\xrightarrow{\text{transition to rank one projector}} \Pi_\psi = |\psi\rangle\langle\psi|$ $\qquad \left\{\begin{array}{l}\text{special case of}\\ \text{density matrix}\end{array}\right.$

2. Mixed state = weighted mixture of pure states

   density matrix $\rho = \rho^\dagger, \quad \rho \geq 0, \quad \operatorname{tr}\rho = 1$

   $\mathcal{D}(\mathcal{H})$ — set of density matrices

3. State $\rho_{XY} \in \mathcal{D}(\mathcal{H}_{XY} = \mathcal{H}_X \otimes \mathcal{H}_Y)$ of composite system is

   - separable if $\rho_{XY} = \sum_k w_k \rho_X^k \otimes \rho_Y^k$ $\qquad w_k \geq 0, \quad \sum_k w_k = 1$
   - entangled otherwise

Quantum mechanics II. Observation and measurement

1. Observation = "click of detector" in subspace $\mathcal{S} \leq \mathcal{H}$ at state ρ

   "detector in $\mathcal{S}$" $\longleftrightarrow$ projector $\Pi_{\mathcal{S}}$

   Gleason's theorem: probability measure $\mu_\rho(\mathcal{S}) = \mathrm{tr}(\rho\Pi_{\mathcal{S}})$

   special case: $\rho = |\psi\rangle\langle\psi|$ and $\mathcal{S} = \mathrm{span}(|\varphi\rangle)$

   $\longrightarrow$ Born's rule: $\mathbf{P}_{\mathrm{Born}} = |\langle\varphi \mid \psi\rangle|^2$

2. Measurement = observation of ρ in eigenspaces of

   Hermitian operator $A = A^\dagger = \sum_k a_k \Pi_{e_k}$ (called *"observable"*)

   - $e_1, e_2, \ldots$ — orthonormal basis of eigenvectors of A
   - $a_1, a_2, \ldots \in \mathbb{R}$ — spectrum of A
   - a_k — result of measurement at click of detector Π_{e_k}
   - $\langle A\rangle_\rho = \mathrm{tr}(\rho A)$ — expectation value of A in state ρ

Quantum mechanics III. Time evolution

1. Evolution = unitary transformation of data between observations at times t and t'
   - $|\psi_{t'}\rangle = U_{t'\,t}\,|\psi_t\rangle$ state vector
   - $\rho_{t'} = U_{t'\,t}\rho_t U^{\dagger}_{t'\,t}$ density matrix

$|\psi_t\rangle$ or ρ_t — state after observation at time t

$|\psi_{t'}\rangle$ or $\rho_{t'}$ — state before observation at time t'

Continuum approximation $\longrightarrow$ Schrödinger equation

Addendum: "Entanglement builds Geometry"

Emergence of geometry within large Hilbert space

1. Decomposing $\mathcal{H}$ into tensor product: $\mathcal{H} = \bigotimes_x \mathcal{H}_x,\ x \in X$
   x's are treated as points (bulks) of geometric space

2. Tensor network is graph G with vertex set X and edges $\{x, y\}$

3. Edges are assigned weights derived from
   a measure of entanglement, typically mutual information:
   $I(\rho_{xy}) = S(\rho_x) + S(\rho_y) - S(\rho_{xy})$, where $S(\rho) = -\operatorname{tr}(\rho \log \rho)$
   Metric is constructed of the weights

4. Approximate isometric embedding of G into smooth metric manifold of as small as possible dimension using algorithms like MDS (multidimensional scaling)

- many models reproduce Bekenstein-Hawking area law (holographic principle)
- Juan Maldacena and Leonard Susskind hypothesized: ER=EPR

Introduction of continuum and differential calculus simplifies problems at the cost of loss of completeness: classification of simple groups
continuous (2 people for ∼6 years) finite (∼100 people for ∼170 years)

Concept of group = abstraction of permutations (one-to-one mappings) of a set

additional assumption: group is	
differentiable manifold	finite
influence on empirical physics	
strong influence	no influence
Lie groups	finite groups ("enormous theorem")
4 infinite series + 5 exceptionals	$16+1+1$ infinite series $+\,26$ sporadic groups
A_n, B_n, C_n, D_n E_6, E_7, E_8, F_4, G_2 Killing, Cartan	$A_n(q), B_n(q), C_n(q), D_n(q), E_6(q), E_7(q), E_8(q), F_4(q), G_2(q)$ ${}^2A_n\left(q^2\right),\ {}^2B_n\left(2^{2n+1}\right),\ {}^2D_n\left(q^2\right),\ {}^3D_4\left(q^3\right)$ ${}^2E_6\left(q^2\right),\ {}^2F_4\left(2^{2n+1}\right),\ {}^2G_2\left(3^{2n+1}\right)$ — of Lee type
	$\mathbb{Z}_p$ — cyclic of prime order A_n — alternating
	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24},\ J_1, J_2, J_3, J_4,\ Co_1, Co_2, Co_3$ $Fi_{22}, Fi_{23}, Fi_{24},\ HS,\ McL,\ He,\ Ru,\ Suz,\ O'N,\ HN$ $Ly,\ Th,\ B,\ M$ — sporadics

Removing infinities from quantum formalism

D. Hilbert: "…the infinite is nowhere to be found in reality.
It neither exists in nature nor provides a legitimate basis for rational thought …"

- Formally

  $\mathrm{U}(n)$ is empirically equivalent to a finite group G

  - $\mathrm{U}(n) \cong \mathrm{Aut}\left(\mathcal{H}_n\right)$
  - $\mathrm{U}(n) \xrightarrow[]{\substack{\text{using a universal set of quantum gates construct}\\ \text{dense in } \mathrm{U}(n) \text{ finitely generated matrix group}}} G_\infty$
  - $G_\infty \xrightarrow[]{\substack{\text{by Maltsev's theorem } G_\infty \text{ is residually finite}\\ \Longrightarrow \text{ rich set of homomorphisms into finite groups}}} \mathsf{G}$

- In essence

  natural assumption: finite groups act at fundamental level, and $\mathrm{U}(n)$'s are only continuum approximations of their unitary representations

- Advantages of finite groups
  - any finite group is a subgroup of a symmetric group
  - any linear representation of a finite group is
    - ⋆ unitary
    - ⋆ subrepresentation of some permutation representation

"Physical" numbers

$\begin{cases} \text{❶ natural numbers } \mathbb{N} = \{0, 1, \ldots\} \text{ — "counters"} \\ \text{❷ roots of unity } \mathrm{r}_k \mid \mathrm{r}_k^k = 1 \text{ — "periodic processes"} \end{cases}$

are sufficient to represent all physically meaningful numbers:

- $\mathbb{Z} = \mathbb{N}[\mathrm{r}_2]$ is extension of semiring $\mathbb{N}$ by 2nd primitive root of unity $\mathrm{r}_2 = \underbrace{\mathrm{e}^{2\pi\mathrm{i}/2} = -1}_{\text{Euler's identity}}$
- ring $\mathbb{N}[\mathrm{r}_k] \xrightarrow{\text{taking fraction field}} k$th cyclotomic field $\mathbb{Q}(\mathrm{r}_k)$
- $\mathbb{Q}(\mathrm{r}_k)$ is dense subfield of $\mathbb{C}$ for $k \geq 3$

Importance for quantum mechanics

- minimal splitting field $F_\mathrm{G} \stackrel{\text{def}}{=\!=}$ minimal extension of $\mathbb{Q}$ that allows to split completely any linear representation of group G into irreps

$F_\mathrm{G} \leq \mathbb{Q}(\mathrm{r}_k)$, k is some divisor of exponent of G

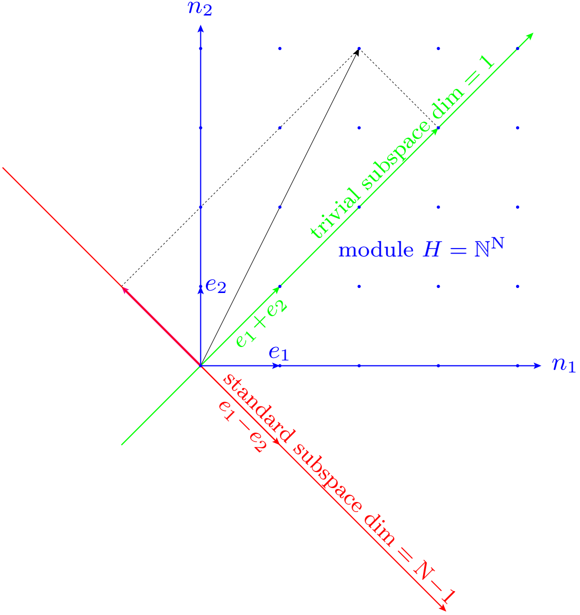
Constructive representations of finite group G

1. Ω is permutation domain for G, $|\Omega| = \mathsf{N}$
2. module $H = \mathbb{N}^{\mathsf{N}}$ over semiring $\mathbb{N}$ with basis Ω
3. Hilbert space: $H \xrightarrow{\mathbb{N}\to\mathbb{Q}(\mathsf{r}_k)} \mathcal{H}$
    - $\mathbb{Q}(\mathsf{r}_k)$ contains splitting field for G
    - H is principal orthant in $\mathbb{Z}^{\mathsf{N}} \subset \mathcal{H}$

Any constructive representation of G can be obtained by projection

of permutation representation of G in module H
onto some subspace of Hilbert space $\mathcal{H}$ over $\mathbb{Q}(\mathsf{r}_k)$

Natural S_N–module $\longrightarrow$ irreducible invariant subspaces



Canonical bases

in trivial subspace

$e_1 + e_2 + \cdots + e_N$

in standard subspace

$e_1 - e_2$

$e_2 - e_3$

$\vdots$

$e_{N-1} - e_N$

Unitary transition between observations

Standard QM: unique unitary evolution $U_{t't} = \mathrm{e}^{-\mathrm{i}H(t'-t)}$

$\downarrow$

Hamiltonian H is obtained from the principle of least action

$\downarrow$

Any extremal principle $\left(\begin{array}{l}\text{with related computational methods:}\\ \text{stationary phase, saddle-point etc}\end{array}\right)$ involves selection of small subset of dominant elements from large set of candidates

$\downarrow$

We assume

all unitary evolutions take part with their weights, but only a small number of dominant evolutions are manifested in observations

Model of quantum evolution (inspired by quantum Zeno effect)

Discrete model | time | Continuum approximation

sequence $t_0, \dots, t_{i-1}, t_i, \dots, t_n$ | continuous interval $[t_0, t_n] \subseteq \mathbb{R}$

unitary evolution operator $U_k = \mathrm{U}(g_k)$, $g_k \in \mathrm{G}$

$\mathrm{G} = \{\mathrm{g}_1, \dots, \mathrm{g}_K\}$, finite group | G, Lie group

$\mathbf{P}_i = \sum_{k=1}^{K} w_{ik} \operatorname{tr}\left(U_k \rho_{i-1} U_k^\dagger \rho_i\right)$, single-step transition probability

$\mathbf{P}_{0\to n} = \prod_{i=1}^{n} \mathbf{P}_i$, probability of trajectory

introduce entropy: same extrema but products $\longrightarrow$ sums

single-step entropy

$\Delta\mathbf{S}_i = -\log \mathbf{P}_i$ | Lagrangian $\mathcal{L}$

entropy of trajectory

$\mathbf{S}_{0\to n} = \sum_{i=1}^{n} \Delta\mathbf{S}_i$ | action $\mathcal{S} = \int \mathcal{L} dt$

Continuum limit of discrete model

Simplifying assumptions

1. probability jump for mixed ρ: $\Delta t \to 0 \Rightarrow \mathsf{P} \to \mathrm{tr}\left(\rho^2\right) < 1$
   $\Longrightarrow$ assume pure states $\rho = |\psi\rangle\langle\psi|$
2. Lie algebra approximation $U \approx \mathbb{1} + \mathrm{i}A$, A is Hermitian matrix
3. linear approximation introducing derivatives $\Delta X \approx \dot{X}\Delta t$

Lagrangian

$$\mathcal{L} = \left\langle \psi \middle| \dot{A}^2 \middle| \psi \right\rangle - \left\langle \psi \middle| \dot{A} \middle| \psi \right\rangle^2 \longleftrightarrow \boxed{\text{dispersion of } \dot{A} \text{ in state } \psi}$$

$$- \mathrm{i}\left(\left\langle \dot{\psi} \middle| \dot{A} \middle| \psi \right\rangle - \left\langle \psi \middle| \dot{A} \middle| \dot{\psi} \right\rangle + 2\left\langle \psi \middle| \dot{A} \middle| \psi \right\rangle \left\langle \psi \mid \dot{\psi} \right\rangle\right) - \left\langle \psi \middle| \dot{\psi} \right\rangle^2$$

Dominant unitary evolutions in symmetric group S_N

Natural vectors: $|n\rangle = \begin{pmatrix} n_1 \\ \vdots \\ n_\mathsf{N} \end{pmatrix}, \ |m\rangle = \begin{pmatrix} m_1 \\ \vdots \\ m_\mathsf{N} \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

Born rule in

1. natural representation: $\mathbf{P}_{\mathrm{nat}}(n,m) = \dfrac{\langle n \,|\, m\rangle^2}{\langle n \,|\, n\rangle \langle m \,|\, m\rangle}$
2. standard representation:

$$\mathbf{P}_{\mathrm{std}}(n,m) = \frac{\left(\langle n \,|\, m\rangle - \frac{1}{\mathsf{N}}\langle n \,|\, 1\rangle\langle 1 \,|\, m\rangle\right)^2}{\left(\langle n \,|\, n\rangle - \frac{1}{\mathsf{N}}\langle n \,|\, 1\rangle\langle 1 \,|\, n\rangle\right)\left(\langle m \,|\, m\rangle - \frac{1}{\mathsf{N}}\langle m \,|\, 1\rangle\langle 1 \,|\, m\rangle\right)}$$

$\left.\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}\right\} \langle n \,|\, m\rangle \Longleftrightarrow n_1, \ldots, n_\mathsf{N}$ and $m_1, \ldots, m_\mathsf{N}$ ordered $\left\{\begin{matrix}\text{identically}\\ \text{oppositely}\end{matrix}\right.$

$\mathbf{P}_*(Un, m)$ is maximized by unitary operator $U = P_m^{-1} P_n$

Permutations P_n, P_m sort n, m $\left\{\begin{matrix}\text{identically for } \mathbf{P}_{\mathrm{nat}}\\ \text{identically or oppositely for } \mathbf{P}_{\mathrm{std}}\end{matrix}\right.$

Energy of permutation

1. Planck's formula: $E = h\nu$
   energy $\simeq$ frequency $= \dfrac{\text{number of detector clicks}}{\text{time interval}}$
   $=$ eigenvalue of Hamiltonian $H = \mathrm{i}\hbar \ln U$
2. Hamiltonian of permutation p of cycle type
   $\left\{ (\ell_1 \text{ — length}, m_1 \text{ — multiplicity}), \ldots, (\ell_K, m_K) \right\}$:

$$H_p = \begin{pmatrix} \mathbb{1}_{m_1} \otimes H_{\ell_1} & & \\ & \ddots & \\ & & \mathbb{1}_{m_K} \otimes H_{\ell_K} \end{pmatrix}$$

$$H_{\ell_k} = \frac{1}{\ell_k} \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \ell_k - 1 \end{pmatrix}$$ — principal Hamiltonian of ℓ_k-cycle

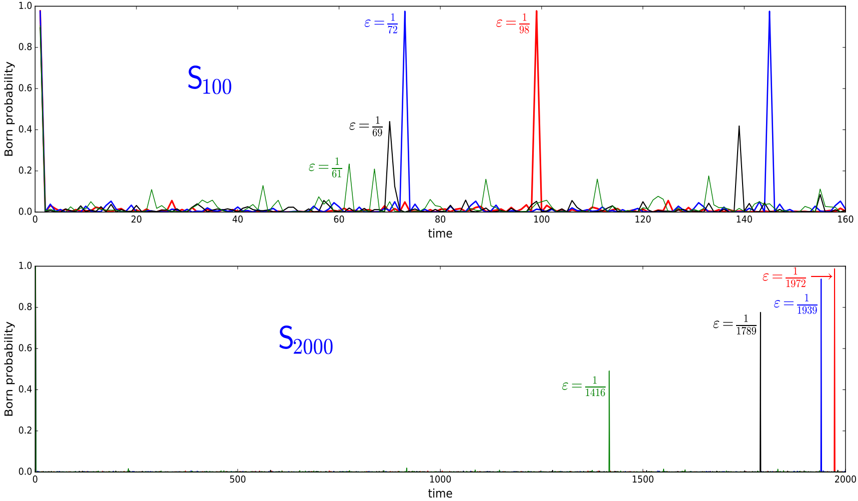
3. Base ("ground state", "zero-point", "vacuum") energy of permutation

$$\boxed{\varepsilon_p = \frac{1}{\max(\ell_1, \ldots, \ell_K)}}$$

Monte Carlo simulation for S_{100} and S_{2000}

$|S_{100}| \approx 9 \times 10^{157}$ $\qquad$ $|S_{2000}| \approx 3 \times 10^{5735}$

Four (red, blue, black, green) randomly generated dominant evolutions



Summary

1. Quantum mechanics $\longleftarrow$
   - permutations of finite sets
   - projections of natural vectors into invariant subspaces
2. Quantum randomness $\longleftarrow$ fundamental impossibility to trace individuality of indistinguishable objects in their evolution
3. Complex numbers in quantum formalism $\longleftarrow$ non-constructive version (metric completion) of cyclotomic numbers
4. Principle of least action $\longleftarrow$ continuum approximation of selection of most likely trajectories
5. Observable behavior of quantum system $\longleftarrow$ dominants among all possible quantum evolutions

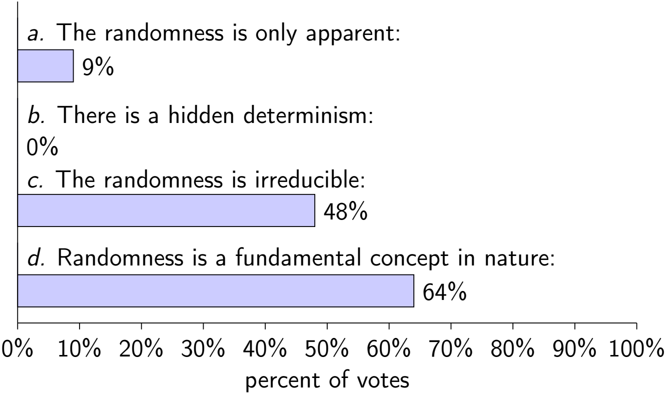
4 Appendix

- Foundational issues of quantum mechanics
- Model of time
- Gauge curvature and quantum uncertainty
- Continuous symmetries as approximations
- Approximate transitivity of symmetric group on quantum states
- Invariant inner products of natural vectors
- Mach–Zehnder interferometer
- Quantum Zeno effect
- Lagrangian for random walk

Physicists believe in fundamental nature of quantum randomness

M. Schlosshauer, J. Kofler, A. Zeilinger[1]
A Snapshot of Foundational Attitudes Toward Quantum Mechanics
Stud. Hist. Phil. Mod. Phys. **44**, 222-230 (2013)

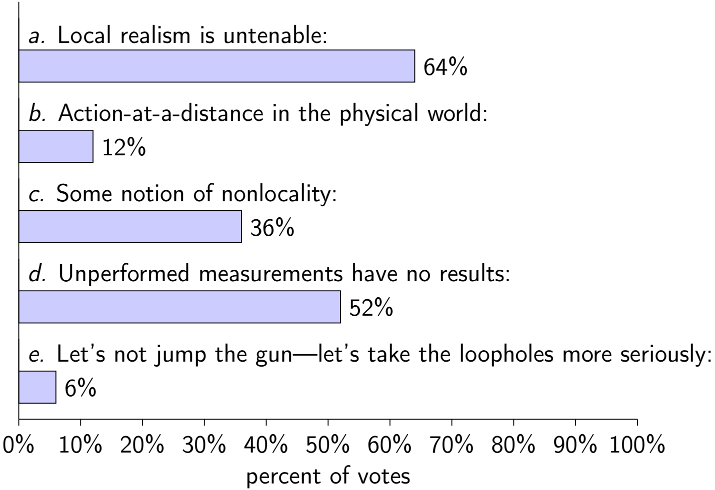
Question 1: What is your opinion about the randomness of individual quantum events (such as the decay of a radioactive atom)?



[1] Anton Zeilinger (Austria) — first realization of quantum teleportation

Physicists do not believe in “local realism”

Question 6: What is the message of the observed violations of Bell’s inequalities?



Epistemic view of quantum behavior

example: Spekkens' toy model (R. W. Spekkens, 2004)

"ontic" states Ω	$\longmapsto$	"epistemic" states are described via rays in $\mathcal{H}$
symmetries $\mathrm{Sym}(\Omega)$		symmetries $\mathsf{Aut}\,(\mathcal{H})$
complete information is not available		partial information is extracted from projections into subspaces of $\mathcal{H}$

we need to specify states mapping: $\Omega \longmapsto \mathcal{H}$

in Spekkens' model artificial knowledge balance principle:

"…for every system, at every time, the amount of knowledge one possesses about the ontic state of the system at that time must equal the amount of knowledge one lacks"

Our assumption about the "loss of ontical information":

quantum randomness arises from fundamental impossibility to trace individuatity of indistinguishable objects in their evolution — only invariant relations are available in observations

Model of time

- Fundamental ("Planck") time: $\mathcal{T} = \mathbb{N}$ (or $\mathbb{Z}$)

- "Empirical time", sequence of "instants of observations":

$$\mathrm{T} = \{t_0, t_1, \ldots, t_{i-1}, t_i, \ldots\}$$

  - simplest assumption: T is a subsequence of $\mathcal{T}$ ie $t_i \in \mathcal{T}$
  - more realistic model: distribution around $t_i \in \mathcal{T}$
    eg binomial distribution

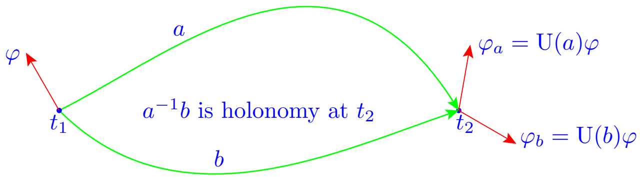
$$K_\sigma\left(\tau - t_i\right) = \frac{(2\sigma)!}{4^\sigma (\sigma - t_i + \tau)!\,(\sigma + t_i - \tau)!}, \quad t_i - \sigma \leq \tau \leq t_i + \sigma$$

    $\sigma = 0$ reproduces simplest assumption

smallest time uncertainty available in physics $\sim 10^{26}$ Planck units

Curvature of gauge connection in continuum approximation

infinitesimal holonomy and quantum uncertainty



$$\langle \varphi_a \mid \varphi_b \rangle = \left\langle \varphi \left| \mathsf{U}\left(a^{-1}b\right) \right| \varphi \right\rangle \xrightarrow{\text{Lie algebra approximation}} \approx \langle \varphi | \mathbb{1} + \mathrm{i}F | \varphi \rangle = 1 + \mathrm{i}\langle \varphi | F | \varphi \rangle$$

normalization: $\left\|(\mathbb{1} + \mathrm{i}F)\varphi\right\|^2 = \langle \varphi | (\mathbb{1} - \mathrm{i}F)(\mathbb{1} + \mathrm{i}F) | \varphi \rangle = 1 + \left\langle \varphi \left| F^2 \right| \varphi \right\rangle$

Probability:

$$\mathbf{P}_{t_1 \to t_2} \approx \frac{1 + \langle \varphi | F | \varphi \rangle^2}{1 + \langle \varphi | F^2 | \varphi \rangle} \xrightarrow{\frac{1}{1+\varepsilon} \approx 1 - \varepsilon} \approx 1 - \left\langle \varphi \left| F^2 \right| \varphi \right\rangle + \langle \varphi | F | \varphi \rangle^2$$

Entropy:

$$\Delta \mathbf{S}_{t_1 \to t_2} = -\ln \mathbf{P}_{t_1 \to t_2} \xrightarrow{\ln(1+\varepsilon) \approx \varepsilon} \approx \left\langle \varphi \left| F^2 \right| \varphi \right\rangle - \langle \varphi | F | \varphi \rangle^2 \equiv (\Delta_\varphi F)^2$$

$\Delta_\varphi F$ — standard deviation $\qquad (\Delta_\varphi F)^2$ — dispersion

Continuous symmetries as approximations I

- Group of integer lattice $\mathbb{Z}^d$:
  $\mathrm{Aut}\left(\mathbb{Z}^d\right) \cong \mathbb{Z}^d \rtimes G_d \qquad G_d \cong \left(\mathbb{Z}_2\right)^d \rtimes \mathsf{S}_d \equiv \mathbb{Z}_2 \wr \mathsf{S}_d$

- Symmetric walk on $\mathbb{Z}^d$ in continuum limit:
  binomial distribution $\xrightarrow[\mathbb{Z} \to \mathbb{R}]{\text{Stirling approximation}}$
  Gauss distribution $\longrightarrow$ product of one-dimensional distributions $\longrightarrow$ heat kernel

$$K\left(t, \vec{x}\right) = \frac{1}{\left(4\pi t\right)^{d/2}} \exp\left(-\frac{x_1^2 + x_2^2 + \cdots + x_d^2}{4t}\right)$$

$t \in \mathbb{R}_{>0} \qquad x_i \in \mathbb{R}$

- Spatial symmetry group of kernel $K\left(t, \vec{x}\right)$
  $\mathbb{R}^d \rtimes \mathrm{O}(d, \mathbb{R})$ — Euclid group = semidirect product of translations and rotations

Continuous symmetries as approximations II

- Asymmetric walk on $\mathbb{Z}$

  k_+, k_- — "right" and "left" step numbers
  $T = k_+ + k_-$ — total number of steps
  $p_+,\ p_-$ — probabilities: $p_+ + p_- = 1$

  Embedding into continuous variables $x, t, v \in \mathbb{R}$:
  $x := k_+ - k_-$ $\quad t := T$ $\quad v := p_+ - p_-$ $\ (-1 \leq v \leq 1)$
  "Drift velocity" v satisfies relativistic velocity addition rule:

  $$w = (u + v) / (1 + uv) \qquad \text{(K.H.Knuth, 2015)}$$

- Continuous approximation of binomial distribution

$$P(x, t) = \sqrt{\frac{2}{\pi (1 - v^2) t}} \exp\left\{-\frac{1}{2t}\left(\frac{x - vt}{\sqrt{1 - v^2}}\right)^2\right\}$$

heat or diffusion or Fokker-Planck equation

$$\frac{\partial P(x, t)}{\partial t} + v \frac{\partial P(x, t)}{\partial x} = \frac{(1 - v^2)}{2} \frac{\partial^2 P(x, t)}{\partial x^2}$$

Continuous symmetries as approximations III

- Approximation with respect to "Hubble time" T_H: $t' \ll T_H$
  Substitutions $t = T_H + t'$ and $x = vT_H + x'$

$$\Downarrow$$

$$P\left(x', t'\right) = \frac{m}{\sqrt{1-v^2}} \exp\left\{-\pi \frac{m^2}{4}\left(\frac{x'-vt'}{\sqrt{1-v^2}}\right)^2\right\} + O\left(\frac{t'}{T_H}\right)$$

$$m = \sqrt{\frac{2}{\pi T_H}}$$

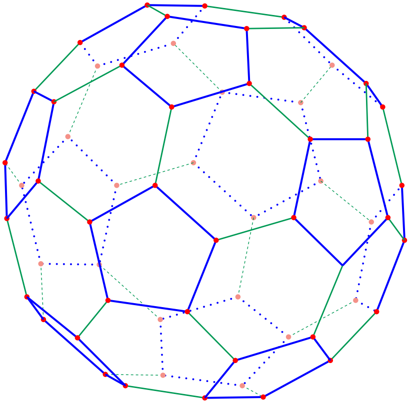
S_N acts "approximately transitively" on pure quantum states

No of random pair $(\|x\rangle, \|y\rangle)$		1	2	3	4
$\mathsf{P}_{\mathrm{std}}(U_{\mathrm{dom}}\,\|x\rangle, \|y\rangle)$	S_{100}	0.977	0.975	0.895	0.905
	S_{2000}	0.999	0.998	0.997	0.998

This resembles transitivity of general unitary group on quantum states

Icosahedral group A_5

order $|A_5| = 60$ exponent $\mathrm{Exp}(A_5) = 30$



- Presentation by 2 generators

$$A_5 = \langle a, b \mid a^5 = b^2 = (ab)^3 = 1 \rangle$$

"physical incarnation": carbon molecule fullerene $C_{60} \cong$ Cayley graph of A_5

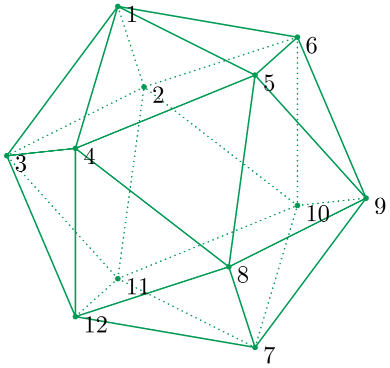
- 5 irreducible representations

$$\mathbf{1}, \mathbf{3}, \mathbf{3}', \mathbf{4}, \mathbf{5}$$

- 3 primitive permutation reps

$$\underline{\mathbf{5}} \cong \mathbf{1} \oplus \mathbf{4}, \ \underline{\mathbf{6}} \cong \mathbf{1} \oplus \mathbf{5}, \ \underline{\mathbf{10}} \cong \mathbf{1} \oplus \mathbf{4} \oplus \mathbf{5}$$

A_5: action on icosahedron



- Action on vertices is imprimitive

  Imprimitivity system

  $$\begin{array}{ccccc} B_1 & \cdots & B_i & \cdots & B_6 \\ \updownarrow & & \updownarrow & & \updownarrow \\ (1,7) & \cdots & (i,i+6) & \cdots & (6,12) \end{array}$$

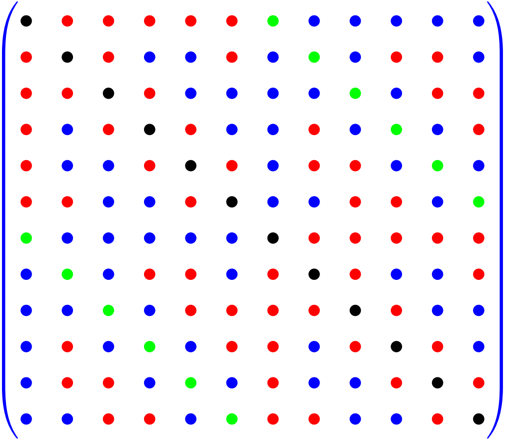
  Blocks are pairs of opposite vertices

- Decomposition into irreps

  $$\underline{\mathbf{12}} \cong \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{5}$$

A_5 on icosahedron: orbitals and centralizer algebra

$\Omega \times \Omega = \{1, \ldots, 12\} \times \{1, \ldots, 12\}$ rank of A_5 on icosahedron $\mathrm{R} = 4$

$\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 =$ 

A_5 on icosahedron: inner product in invariant subspaces

Invariant forms in invariant subspaces obtained
by solution of 4 systems of 4 linear equations

$$
\begin{aligned}
\mathcal{B}_{\mathbf{1}} =& \frac{1}{12}\left(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4\right)\\
\mathcal{B}_{\mathbf{3}} =& \frac{1}{4}\left(\mathcal{A}_1 - \mathcal{A}_2 - \frac{1+2\mathrm{r}^2+2\mathrm{r}^3}{5}\mathcal{A}_3 + \frac{1+2\mathrm{r}^2+2\mathrm{r}^3}{5}\mathcal{A}_4\right)\\
\mathcal{B}_{\mathbf{3}'} =& \frac{1}{4}\left(\mathcal{A}_1 - \mathcal{A}_2 + \frac{1+2\mathrm{r}^2+2\mathrm{r}^3}{5}\mathcal{A}_3 - \frac{1+2\mathrm{r}^2+2\mathrm{r}^3}{5}\mathcal{A}_4\right)\\
\mathcal{B}_{\mathbf{5}} =& \frac{5}{12}\left(\mathcal{A}_1 + \mathcal{A}_2 - \frac{1}{5}\mathcal{A}_3 - \frac{1}{5}\mathcal{A}_4\right)
\end{aligned}
$$

r is 5th primitive root of unity

A_5 on icosahedron: scalar products of projections of natural amplitudes

$n = (n_1, \ldots, n_{12})^T$, $m = (m_1, \ldots, m_{12})^T$ — natural vectors
Ψ_α, Φ_α — projections of n, m onto invariant subspaces

1. $\langle \Phi_1 | \Psi_1 \rangle = \frac{1}{12} \{\mathcal{A}_1(m,n) + \mathcal{A}_2(m,n) + \mathcal{A}_3(m,n) + \mathcal{A}_4(m,n)\} \equiv \frac{1}{12} L(m) L(n)$
   invariant $L(n) = \sum_{k=1}^{12} n_k$ is "total number of particles"
2. $\langle \Phi_{\mathbf{3}\oplus\mathbf{3}'} \mid \Psi_{\mathbf{3}\oplus\mathbf{3}'} \rangle = \frac{1}{2} \{\mathcal{A}_1(m,n) - \mathcal{A}_2(m,n)\}$
   1. $\langle \Phi_{\mathbf{3}} \mid \Psi_{\mathbf{3}} \rangle = \frac{1}{4} \left\{\mathcal{A}_1(m,n) - \mathcal{A}_2(m,n) + \frac{\sqrt{5}}{5}(\mathcal{A}_3(m,n) - \mathcal{A}_4(m,n))\right\}$
   2. $\langle \Phi_{\mathbf{3}'} | \Psi_{\mathbf{3}'} \rangle = \frac{1}{4} \left\{\mathcal{A}_1(m,n) - \mathcal{A}_2(m,n) - \frac{\sqrt{5}}{5}(\mathcal{A}_3(m,n) - \mathcal{A}_4(m,n))\right\}$

   - irrationality is consequence of imprimitivity: one can not move icosahedron vertex without simultaneous movement of its opposite $\Longrightarrow$ only combination $\mathbf{3} \oplus \mathbf{3}'$ makes sense
3. $\langle \Phi_{\mathbf{5}} \mid \Psi_{\mathbf{5}} \rangle = \frac{5}{12} \left\{\mathcal{A}_1(m,n) + \mathcal{A}_2(m,n) - \frac{1}{5}(\mathcal{A}_3(m,n) + \mathcal{A}_4(m,n))\right\}$

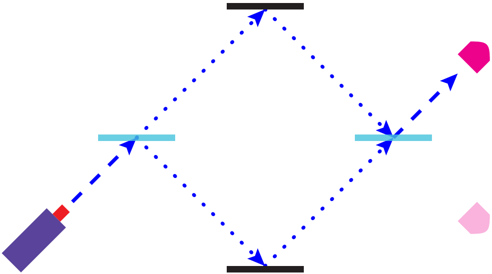
Mach–Zehnder interferometer

Beam-splitter S: $\begin{aligned} |\nearrow\rangle &\rightarrow \frac{1}{\sqrt{2}}\left(|\nearrow\rangle + \mathrm{i}\,|\searrow\rangle\right) \\ |\searrow\rangle &\rightarrow \frac{1}{\sqrt{2}}\left(|\searrow\rangle + \mathrm{i}\,|\nearrow\rangle\right) \end{aligned}$ $\quad S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix}$

Mirror M: $\begin{aligned} |\nearrow\rangle &\rightarrow \mathrm{i}\,|\searrow\rangle \\ |\searrow\rangle &\rightarrow \mathrm{i}\,|\nearrow\rangle \end{aligned}$ $\quad M = \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$ $\quad M = S^2$

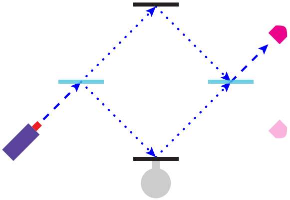
S generates group $\mathbb{Z}_8$

Scheme



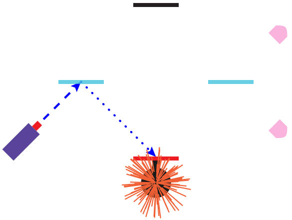
implements evolution $SMS\,|\nearrow\rangle = S^4\,|\nearrow\rangle = -\,|\nearrow\rangle$

Elitzur–Vaidman interaction-free measurements. Penrose bomb tester



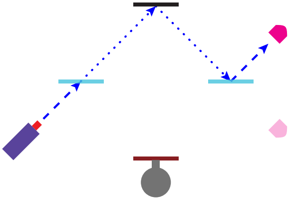
$$|\nearrow\rangle \xrightarrow{SMS} -|\nearrow\rangle \quad \mathbf{P}=1$$

testing dud bomb



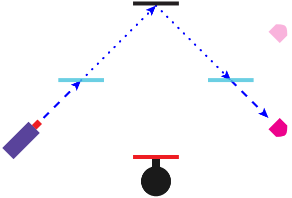
$$|\nearrow\rangle \xrightarrow{\Pi_{\searrow}S} \frac{\mathrm{i}}{\sqrt{2}}|\searrow\rangle \quad \mathbf{P}=\frac{1}{2}$$

good bomb went off



$$|\nearrow\rangle \xrightarrow{\Pi_{\nearrow}S\,M\,\Pi_{\nearrow}S} -\frac{1}{2}|\nearrow\rangle \quad \mathbf{P}=\frac{1}{4}$$

bomb remains untested



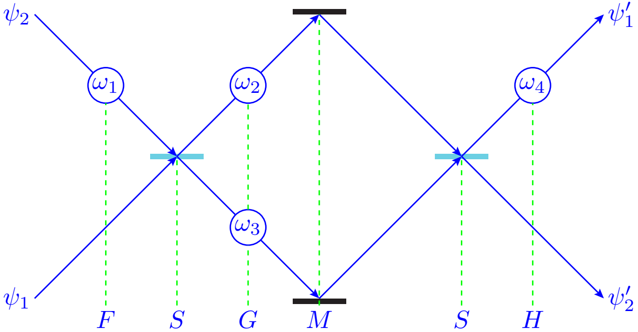
$$|\nearrow\rangle \xrightarrow{\Pi_{\searrow}S\,M\,\Pi_{\nearrow}S} \frac{\mathrm{i}}{2}|\searrow\rangle \quad \mathbf{P}=\frac{1}{4}$$

bomb is good and intact

Mach–Zehnder interferometer implements any one-qubit gate

$\dim \mathrm{U}(2) = 4 \Longrightarrow$ need to add 4 phase shifters $\omega_1, \omega_2, \omega_3, \omega_4$
to implement arbitrary unitary 2×2 matrix U

one of 16 possibilities:



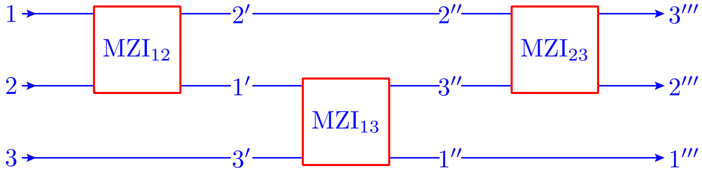
$$U_{\mathrm{MZI}} = HSMGSF$$

$$F = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\omega_1} \end{bmatrix} \quad S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad G = \begin{bmatrix} e^{i\omega_2} & 0 \\ 0 & e^{i\omega_3} \end{bmatrix} \quad M = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad H = \begin{bmatrix} e^{i\omega_4} & 0 \\ 0 & 1 \end{bmatrix}$$

MZI implementation of arbitrary matrix $U \in \mathrm{U}(n)$

$$U = \prod_{1 \le i < j \le n} \mathbb{1}_{\{1,\ldots,\widehat{i},\ldots,\widehat{j},\ldots,n\}} \oplus U_{\mathrm{MZI}_{ij}}$$

- sequence of $\dfrac{n(n-1)}{2}$ Mach–Zehnder interferometers corresponding to two-dimensional subspaces of $\mathcal{H}_n$



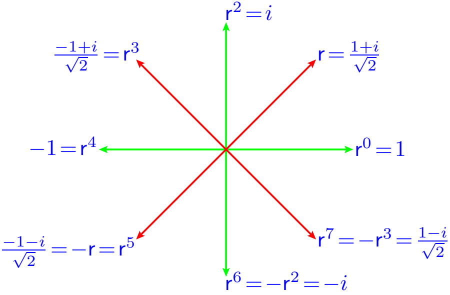
- $\dim \mathrm{U}(n) = n^2 \Longrightarrow \begin{cases} \text{excess in number of parameters} \\ 4\dfrac{n(n-1)}{2} - n^2 = n^2 - 2n \end{cases}$

a more economical scheme in:

M. Reck, A. Zeilinger, H. J. Bernstein, P. Bertani "Experimental Realization of Any Discrete Unitary Operator" *Phys. Rev. Lett.* **73** (1994) 58

Constructive view on balanced Mach–Zehnder interferometer I

- Mirror is square of beam-splitter: $M = S^2$
  $\Longrightarrow$ any sequence of MZI's can be described by degrees of S
- S generates cyclic group $\mathbb{Z}_8$
  - Cyclotomic polynomial $\Phi_8(\mathsf{r}) = 1 + \mathsf{r}^4$
  - primitive and nonprimitive roots of unity



  - smallest degree of faithful permutation action $= 8$

Constructive view on balanced Mach–Zehnder interferometer II

- embedding into permutations
  - $S \longleftrightarrow g = (1,2,3,4,5,6,7,8)$
  - representation in 8D module of natural vectors $\mathbb{N}^8$
    $N = (n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)^T \in \mathbb{N}^8$
  - similarity transformation over 8th cyclotomic field:

$$
\begin{array}{ccc}
P(g) & \longrightarrow & S(g) = T^{-1} P(g)\, T \\
\| & & \| \\
\begin{bmatrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
& \longrightarrow &
\begin{bmatrix}
1 & \cdot & \cdot \\
\cdot & A & \cdot \\
\cdot & \cdot & \overbrace{\begin{bmatrix} \frac{\mathrm{r} - \mathrm{r}^3}{2} & \frac{\mathrm{r} + \mathrm{r}^3}{2} \\ \frac{\mathrm{r} + \mathrm{r}^3}{2} & \frac{\mathrm{r} - \mathrm{r}^3}{2} \end{bmatrix}}^{\text{beam splitter } S}
\end{bmatrix}
\end{array}
$$

r is 8th primitive root of unity $\qquad A = \operatorname{diag}\left(-1, \mathrm{r}^2, -\mathrm{r}^2, \mathrm{r}^3, -\mathrm{r}\right)$

Constructive view on balanced Mach–Zehnder interferometer III

- quantum amplitude as projection of N into "splitter" subspace

$$|\psi\rangle = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-\mathrm{r}^3\left(n_1+n_3-n_5-n_7\right)+\left(1-\mathrm{r}^2\right)\left(n_2-n_6\right)\\ \mathrm{r}\left(n_1-n_3-n_5+n_7\right)+\left(1+\mathrm{r}^2\right)\left(-n_4+n_8\right)\end{bmatrix}$$

  - $|\psi\rangle$ represents with arbitrary precision any point on the Bloch sphere — complex projective line $\mathbb{C}P^1$
  - minuses and denominators can be eliminated

    $\Longrightarrow$ quantum state can be expressed in terms of
    natural numbers and roots of unity

Quantum Zeno effect ("Turing paradox")

Dynamics of quantum system under frequent observations

- frequent observations may $\begin{cases} \text{stop (slow down) quantum evolution} \\ \text{— quantum Zeno effect} \\ \text{force prescribed evolution} \\ \text{— "anti-Zeno effect"} \end{cases}$
- probability to observe initial state $p_Z(t) = \left|\left\langle \psi_0 \,\middle|\, e^{-\mathrm{i}Ht} \,\middle|\, \psi_0 \right\rangle\right|^2$
- short-time expansion $p_Z(t) = 1 - t^2/\boldsymbol{\tau}_Z^2 + \mathrm{O}\left(t^4\right)$
- Zeno time $\boldsymbol{\tau}_Z^{-2} = \left\langle \psi_0 \,\middle|\, H^2 \,\middle|\, \psi_0 \right\rangle - \left\langle \psi_0 \,\middle|\, H \,\middle|\, \psi_0 \right\rangle^2$

Zeno dynamics in our framework

- sequence of observations $\Pi_{\psi_{t_0}}, \Pi_{\psi_{t_1}}, \ldots, \Pi_{\psi_{t_N}}$ of the same state: $\psi_{t_0} = \psi_{t_1} = \cdots = \psi_{t_N} \equiv \psi_0$
- $t_0 = 0,\ t_N = T,\ t_i - t_{i-1} = T/N$ — equidistant observation times
- $\mathbf{P}_{\psi_{t_{i-1}} \to \psi_{t_i}} \approx 1 - \frac{1}{N^2}\left(\frac{T}{\tau_Z}\right)^2$ — short-time approximation
- $\Delta\mathbf{S}_{\psi_{t_{i-1}} \to \psi_{t_i}} \approx \frac{1}{N^2}\left(\frac{T}{\tau_Z}\right)^2$ — approximated one-step entropy
- entropy of trajectory

$$\mathbf{S}_{\psi_{t_0} \to \cdots \to \psi_{t_N}} = \sum_{i=1}^{N} \Delta\mathbf{S}_{\psi_{t_{i-1}} \to \psi_{t_i}} \approx \frac{1}{N}\left(\frac{T}{\tau_Z}\right)^2 \xrightarrow{N \to \infty} 0$$

- probability of trajectory tends to 1 — the essence of Zeno effect

$$\mathbf{P}_{\psi_{t_0} \to \cdots \to \psi_{t_N}} \approx \prod_{i=1}^{N}\left(1 - \frac{1}{N^2}\left(\frac{T}{\tau_Z}\right)^2\right) \xrightarrow{N \to \infty} \mathrm{e}^0 = 1$$

Zeno dynamics for unbalanced beam splitter

$$S_N = \frac{1}{2}\begin{bmatrix} \mathrm{r} + \mathrm{r}^{N-1} & \mathrm{r} - \mathrm{r}^{N-1} \\ \mathrm{r} - \mathrm{r}^{N-1} & \mathrm{r} + \mathrm{r}^{N-1} \end{bmatrix} \quad \text{probability of} \begin{cases} \text{passage} & = \frac{1}{2} + \frac{\mathrm{r}^2 + \mathrm{r}^{N-2}}{4} \\ \text{reflection} & = \frac{1}{2} - \frac{\mathrm{r}^2 + \mathrm{r}^{N-2}}{4} \end{cases}$$

r is Nth primitive root of unity

S_N generates $2D$ representation of $\mathbb{Z}_N$

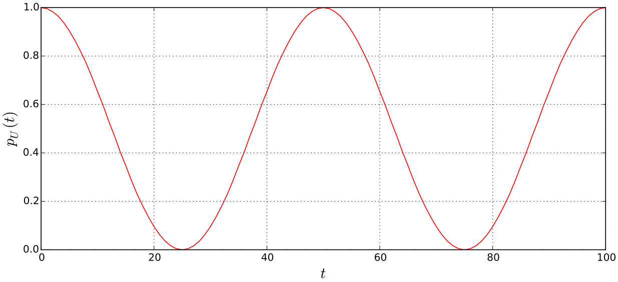


Figure: $p_U(t)$ vs t for operator $U = S_{100} \in \mathrm{U}(\mathbb{Z}_{100})$

Example: Lagrangian from combinatorics I

$$P_{k_1,k_2,t} = \frac{t!}{k_1!k_2!}\alpha_1^{k_1}\alpha_2^{k_2} \quad — \quad \begin{cases} (1+1)\text{D random walk} \\ k_1+k_2=t,\ \alpha_1+\alpha_2=1 \end{cases}$$

$$\Big\downarrow \begin{array}{l} x := k_1 - k_2 \\ v := \alpha_1 - \alpha_2 \quad \text{— “drift velocity”} \quad -1 \leq v \leq 1 \end{array}$$

$$P(x,t) = \frac{t!}{\left(\frac{t+x}{2}\right)!\left(\frac{t-x}{2}\right)!}\left(\frac{1+v}{2}\right)^{\frac{t+x}{2}}\left(\frac{1-v}{2}\right)^{\frac{t-x}{2}}$$

- fundamental (“Planck”) time $[0, 1, \ldots, T]$
- microscopic time (“observation times”) $[t_0 = 0, \ldots, t_{i-1}, t_i, \ldots, t_n = T]$
- observed values $[X_0, \ldots, X_{i-1}, X_i, \ldots, X_n]$

$\Delta t_i = t_i - t_{i-1}, \qquad 1 \ll \Delta t_i \ll T$

$\Delta X_i = X_i - X_{i-1}, \qquad v_i$ — drift velosity in $[t_{i-1}, t_i]$

Example: Lagrangian from combinatorics II

$$\mathbf{P}_{X_{i-1}\to X_i} = \frac{\Delta t_i!}{\left(\frac{\Delta t_i+\Delta X_i}{2}\right)!\left(\frac{\Delta t_i-\Delta X_i}{2}\right)!}\left(\frac{1+v_i}{2}\right)^{\frac{\Delta t_i+\Delta X_i}{2}}\left(\frac{1-v_i}{2}\right)^{\frac{\Delta t_i-\Delta X_i}{2}}$$

$$\Delta\mathbf{S}_{X_{i-1}\to X_i} = -\ln \mathbf{P}_{X_{i-1}\to X_i}$$

1. Stirling approximation: $\ln n! \approx n\ln n - n$
2. 2nd order expansion at stationary point $\Delta X_i^* = v_i \Delta t_i$
3. continuum approximation $X_i \to x(t)$, $v_i \to v(t)$

$$\Delta X_i \approx \dot{x}(t)\,\Delta t_i$$

$$\Delta\mathbf{S}_{X_{i-1}\to X_i} \approx \frac{1}{2}\left(\frac{\dot{x}(t)-v}{\sqrt{1-v^2}}\right)^2 \Delta t_i \Longrightarrow \text{Lagrangian } \mathcal{L} = \left(\frac{\dot{x}(t)-v}{\sqrt{1-v^2}}\right)^2$$

Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = 0 \;\Longrightarrow\; \ddot{x}\left(1-v^2\right) + 2\dot{x}v\frac{\partial v}{\partial t} - \left(1+v^2\right)\frac{\partial v}{\partial t} = 0$$