GRID and Quanputers

Nugzar Makhaldiani

JINR Dubna, mnv@jinr.ru

Talk presented at the 7th International Conference "Distributed Computing and Grid-technologies in Science and Education"
Russia, Dubna, LIT JINR, 4-8 Jule 2016

If you are only a poet,
You are not even that.
(Piet Hein)

In this talk, we consider a general method of **hamiltonization** of the dynamical systems. In the case of the discrete dynamical systems, we define a family of time-invertible dynamical systems and their linear extensions - quanputers, which contains contemporary models of quantum computers. Then we describe GRID as discrete dynamical system and suggest some contemporary and future modifications of GRID according **quanputer technologies.**

We say that we find New Physics when either we find a phenomenon which is forbidden by SM in principal - this is the qualitative level of New physics - or we find significant deviation between precision calculations in SM of an observable quantity and corresponding experimental value.
In 1900, the British physicist Lord Kelvin is said to have pronounced: "There is nothing new to be discovered in physics now. All that remains is more and more precise measurement." Within three decades, quantum mechanics and Einstein's theory of relativity had revolutionized the field. Today, no physicist would dare assert that our physical knowledge of the universe is near completion. To the contrary, each new discovery seems to unlock a Pandora's box of even bigger, even deeper physics questions.

Let us consider a general dynamical system described by the following system of the ordinary differential equations [Arnold, 1978]

$$\dot{x}_n = v_n(x),\ 1 \leq n \leq N, \tag{1}$$

$\dot{x}_n$ stands for the total derivative with respect to the parameter t.
When the number of the degrees of freedom is even, and

$$v_n(x) = \varepsilon_{nm} \frac{\partial H_0}{\partial x_m},\ 1 \leq n, m \leq 2M, \tag{2}$$

the system (1) is Hamiltonian one and can be put in the form

$$\dot{x}_n = \{x_n, H_0\}_0, \tag{3}$$

where the Poisson bracket is defined as

$$\{A, B\}_0 = \varepsilon_{nm} \frac{\partial A}{\partial x_n} \frac{\partial B}{\partial x_m} = A \frac{\overleftarrow{\partial}}{\partial x_n} \varepsilon_{nm} \frac{\overrightarrow{\partial}}{\partial x_m} B, \tag{4}$$

and summation rule under repeated indices has been used.

Let us consider the following Lagrangian

$$L = (\dot{x}_n - v_n(x))\psi_n \quad (5)$$

and the corresponding equations of motion

$$\dot{x}_n = v_n(x), \dot{\psi}_n = -\frac{\partial v_m}{\partial x_n}\psi_m. \quad (6)$$

The system (6) extends the general system (1) by linear equation for the variables ψ. The extended system can be put in the Hamiltonian form [Makhaldiani, Voskresenskaya, 1997]

$$\dot{x}_n = \{x_n, H_1\}_1, \dot{\psi}_n = \{\psi_n, H_1\}_1, \quad (7)$$

where first level (order) Hamiltonian is

$$H_1 = v_n(x)\psi_n \quad (8)$$

and (first level) bracket is defined as

$$\{A, B\}_1 = A(\frac{\overleftarrow{\partial}}{\partial x_n}\frac{\overrightarrow{\partial}}{\partial \psi_n} - \frac{\overleftarrow{\partial}}{\partial \psi_n}\frac{\overrightarrow{\partial}}{\partial x_n})B. \quad (9)$$

Note that when the Grassmann grading [Berezin, 1987] of the conjugated variables x_n and ψ_n are different, the bracket (9) is known as Buttin bracket[Buttin, 1996].

In the Faddeev-Jackiw formalism [Faddeev, Jackiw, 1988] for the Hamiltonian treatment of systems defined by first-order Lagrangians, i.e. by a Lagrangian of the form

$$L = f_n(x)\dot{x}_n - H(x), \tag{10}$$

motion equations

$$f_{mn}\dot{x}_n = \frac{\partial H}{\partial x_m}, \tag{11}$$

for the regular structure function f_{mn}, can be put in the explicit hamiltonian (Poisson; Dirac) form

$$\dot{x}_n = f^{-1}_{nm}\frac{\partial H}{\partial x_m} = \{x_n, x_m\}\frac{\partial H}{\partial x_m} = \{x_n, H\}, \tag{12}$$

where the fundamental Poisson (Dirac) bracket is

$$\{x_n, x_m\} = f^{-1}_{nm},\ f_{mn} = \partial_m f_n - \partial_n f_m. \tag{13}$$

The system (6) is an important example of the first order regular hamiltonian systems. Indeed, in the new variables,

$$y_n^1 = x_n, y_n^2 = \psi_n, \tag{14}$$

lagrangian (5) takes the following first order form

$$\begin{aligned} L &= (\dot{x}_n - v_n(x))\psi_n \Rightarrow \frac{1}{2}(\dot{x}_n\psi_n - \dot{\psi}_n x_n) - v_n(x)\psi_n \\ &= \frac{1}{2} y_n^a \varepsilon^{ab} \dot{y}_n^b - H(y) \\ &= f_n^a(y)\dot{y}_n^a - H(y), f_n^a = \frac{1}{2} y_n^b \varepsilon^{ba}, H = v_n(y^1) y_n^2, \\ f_{nm}^{ab} &= \frac{\partial f_m^b}{\partial y_n^a} - \frac{\partial f_n^a}{\partial y_m^b} = \varepsilon^{ab}\delta_{nm}; \end{aligned} \tag{15}$$

corresponding motion equations and the fundamental Poisson bracket are

$$\dot{y}_n^a = \varepsilon_{ab}\delta_{nm}\frac{\partial H}{\partial y_m^b} = \{y_n^a, H\}, \{y_n^a, y_m^b\} = \varepsilon_{ab}\delta_{nm}. \tag{16}$$

To the canonical quantization of this system corresponds

$$[\hat{y}_n^a, \hat{y}_m^b] = i\hbar\varepsilon_{ab}\delta_{nm},\ \hat{y}_n^1 = y_n^1,\ \hat{y}_n^2 = -i\hbar\frac{\partial}{\partial y_n^1} \tag{17}$$

In this quantum theory, classical part, motion equations for y_n^1, remain classical.

Nabu – Babylonian God
of Wisdom and Writing.

The Hamiltonian mechanics (HM) is in the fundamentals of mathematical description of the physical theories [Faddeev, Takhtajan, 1990]. But HM is in a sense blind; e.g., it does not make a difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) [Sinai, 1993] and (super)integrable Hamiltonian systems (with maximal number of the integrals of motion).
Nabu mechanics (NM) [Nambu, 1973, Whittaker, 1927] is a proper generalization of the HM, which makes the difference between dynamical systems with different numbers of integrals of motion explicit (see, e.g.[Makhaldiani, 2007]).

In the canonical formulation, the equations of motion of a physical system are defined via a Poisson bracket and a Hamiltonian, [Arnold, 1978]. In Nambu's formulation, the Poisson bracket is replaced by the Nambu bracket with $n+1, n \geq 1$, slots. For $n = 1$, we have the canonical formalism with one Hamiltonian. For $n \geq 2$, we have Nambu-Poisson formalism, with n Hamiltonians, [Nambu, 1973], [Whittaker, 1927].

The system of N vortexes can be described by the following system of differential equations, [Aref, 1983, Meleshko,Konstantinov, 1993]

$$\dot{z}_n = i \sum_{m \neq n}^{N} \frac{\gamma_m}{z_n^* - z_m^*}, \quad 1 \leq n \leq N, \tag{18}$$

where $z_n = x_n + iy_n$ are complex coordinate of the centre of n-th vortex, for $N = 3$, and the quantities

$$\begin{aligned} u_1 &= ln|z_2 - z_3|^2, \\ u_2 &= ln|z_3 - z_1|^2, \\ u_3 &= ln|z_1 - z_2|^2 \end{aligned} \tag{19}$$

reduce to the following system

$$\begin{aligned} \dot{u}_1 &= \gamma_1(e^{u_2} - e^{u_3}), \\ \dot{u}_2 &= \gamma_2(e^{u_3} - e^{u_1}), \\ \dot{u}_3 &= \gamma_3(e^{u_1} - e^{u_2}), \end{aligned} \tag{20}$$

The system (20) has two integrals of motion

$$H_1 = \sum_{i=1}^{3} \frac{e^{u_i}}{\gamma_i}, H_2 = \sum_{i=1}^{3} \frac{u_i}{\gamma_i}$$

and can be presented in the Nambu–Poisson form, [Makhaldiani, 1997,2]

$$\dot{u}_i = \omega_{ijk} \frac{\partial H_1}{\partial u_j} \frac{\partial H_2}{\partial u_k} = \{x_i, H_1, H_2\} = \omega_{ijk} \frac{e^{u_j}}{\gamma_j} \frac{1}{\gamma_k},$$

where

$$\omega_{ijk} = \epsilon_{ijk} \rho, \rho = \gamma_1 \gamma_2 \gamma_3$$

and the Nambu–Poisson bracket of the functions A, B, C on the three-dimensional phase space is

$$\{A, B, C\} = \omega_{ijk} \frac{\partial A}{\partial u_i} \frac{\partial B}{\partial u_j} \frac{\partial C}{\partial u_k}. \qquad (21)$$

This system is superintegrable: for $N = 3$ degrees of freedom, we have maximal number of the integrals of motion $N - 1 = 2$.

The reduction of the dimensionless couplings in GUTs is achieved by searching for RD integrals of motion-renormdynamic invariant (RDI) relations among them holding beyond the unification scale. Finiteness results from the fact that there exist RDI relations among dimensional couplings that guarantee the vanishing of all beta-functions in certain GUTs even to all orders. In this case the number of the independent motion integrals N is equal to the number of the coupling constants. Note that in superintegrable dynamical systems the number of the integrals is $\leq N-1$, so the RD of the finite field theories is trivial, coupling constants do not run, the have fixed values, the renormdynamics is more than superintegrable, it is hyperintegrable. Developments in the soft supersymmetry breaking sector of GUTs and FUTs lead to exact RDI relations, i.e. reduction of couplings, in this dimensionful sector of the theory, too. Based on the above theoretical framework phenomenologically consistent FUTs have been constructed. The main goal expected from a unified description of interactions by the particle physics community is to understand the present day large number of free parameters of the SM in terms of a few fundamental ones. In other words, to achieve reduction of couplings at a more fundamental level.

As an example of the infinite dimensional Nambu-Poisson dynamics, let me conside the following extension of Schr$\ddot{o}$dinger quantum mechanics [Makhaldiani, 2000]

$$iV_t = \Delta V - \frac{V^2}{2}, \tag{22}$$

$$i\psi_t = -\Delta\psi + V\psi. \tag{23}$$

An interesting solution to the equation for the potential (22) is

$$V = \frac{4(4-d)}{r^2}, \tag{24}$$

where d is the dimension of the spase. In the case of $d = 1$, we have the potential of conformal quantum mechanics.
The variational formulation of the extended quantum theory, is given by the following Lagrangian

$$L = (iV_t - \Delta V + \frac{1}{2}V^2)\psi. \tag{25}$$

The momentum variables are

$$P_v = \frac{\partial L}{\partial V_t} = i\psi, P_\psi = 0. \tag{26}$$

As Hamiltonians of the Nambu-theoretic formulation, we take the following integrals of motion

$$\begin{aligned} H_1 &= \int d^dx(\Delta V - \frac{1}{2}V^2)\psi, \\ H_2 &= \int d^dx(P_v - i\psi), \\ H_3 &= \int d^dxP_\psi. \end{aligned} \tag{27}$$

We invent unifying vector notation, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) = (\psi, P_\psi, V, P_v)$. Then it may be verified that the equations of the extended quantum theory can be put in the following Nambu-theoretic form

$$\phi_t(x) = \{\phi(x), H_1, H_2, H_3\}, \tag{28}$$

where the bracket is defined as

$$\{A_1, A_2, A_3, A_4\} = i\varepsilon_{ijkl} \int \frac{\delta A_1}{\delta \phi_i(y)} \frac{\delta A_2}{\delta \phi_j(y)} \frac{\delta A_3}{\delta \phi_k(y)} \frac{\delta A_4}{\delta \phi_l(y)} dy$$
$$= i \int \frac{\delta(A_1, A_2, A_3, A_4)}{\delta(\phi_1(y), \phi_2(y), \phi_3(y), \phi_4(y))} dy = idet(\frac{\delta A_k}{\delta \phi_l}). \tag{29}$$

The basic building blocks of M theory are membranes and $M5-$branes. Membranes are fundamental objects carrying electric charges with respect to the 3-form C-field, and $M5$-branes are magnetic solitons. The Nambu-Poisson 3-algebras appear as gauge symmetries of superconformal Chern-Simons nonabelian theories in $2+1$ dimensions with the maximum allowed number of $N=8$ linear supersymmetries.
The Bagger and Lambert [Bagger, Lambert, 2007] and, Gustavsson [Gustavsson, 2007] (BLG) model is based on a 3-algebra,

$$[T^a, T^b, T^c] = f^{abc}_d T^d \tag{30}$$

where T^a, are generators and f_{abcd} is a fully anti-symmetric tensor.

Given this algebra, a maximally supersymmetric Chern-Simons lagrangian is:

$$L = L_{CS} + L_{matter},$$
$$L_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}(f_{abcd}A^{ab}_{\mu}\partial_{\nu}A^{cd}_{\lambda} + \frac{2}{3}f_{cdag}f^{g}_{efb}A^{ab}_{\mu}A^{cd}_{\nu}A^{ef}_{\lambda}), \quad (31)$$

$$L_{matter} = \frac{1}{2}B^{Ia}_{\mu}B^{\mu I}_{a} - B^{Ia}_{\mu}D^{\mu}X^{I}_{a}$$
$$+\frac{i}{2}\bar{\psi}^{a}\Gamma^{\mu}D_{\mu}\psi_{a} + \frac{i}{4}\bar{\psi}^{b}\Gamma_{IJ}x^{I}_{c}x^{J}_{d}\psi_{a}f^{abcd}$$
$$-\frac{1}{12}tr([X^{I},X^{J},X^{K}][X^{I},X^{J},X^{K}]),\ I = 1,2,...,8, \quad (32)$$

where A^{ab}_{μ} is gauge boson, ψ^{a} and $X^{I} = X^{I}_{a}T^{a}$ matter fields. If $a = 1,2,3,4$, then we can obtain an $SO(4)$ gauge symmetry by choosing $f_{abcd} = f\varepsilon_{abcd}, f$ being a constant. It turns out to be the only case that gives a gauge theory with manifest unitarity and $N = 8$ supersymmetry.

The action has the first order form so we can use the formalism of the first section. The motion equations for the gauge fields

$$f^{nm}_{abcd}\dot{A}^{cd}_{m}(t,x) = \frac{\delta H}{\delta A^{ab}_{n}(t,x)}, f^{nm}_{abcd} = \varepsilon^{nm} f_{abcd} \tag{33}$$

take canonical form

$$\dot{A}^{ab}_{n} = f^{abcd}_{nm}\frac{\delta H}{\delta A^{cd}_{m}} = \{A^{ab}_{n}, A^{cd}_{m}\}\frac{\delta H}{\delta A^{cd}_{m}} = \{A^{ab}_{n}, H\},$$
$$\{A^{ab}_{n}(t,x), A^{cd}_{m}(t,y)\} = \varepsilon_{nm} f^{abcd}\delta^{(2)}(x-y) \tag{34}$$

The quasi-classical description of the motion of a relativistic (nonradiating) point particle with spin in accelerators and storage rings includes the equations of orbit motion

$$\begin{aligned} &\dot{x}_n = f_n(x),\ f_n(x) = \varepsilon_{nm}\partial_m H,\ n,m = 1,2,...,6; \\ &x_n = q_n,\ x_{n+3} = p_n,\ \varepsilon_{n,n+3} = 1,\ n = 1,2,3; \\ &H = e\Phi + c\sqrt{\wp^2 + m^2c^2},\ \wp_n = p_n - \frac{e}{c}A_n \end{aligned} \tag{35}$$

and Thomas-BMT equations
[Tomas, 1927, Bargmann, Michel,Telegdi, 1959] of classical spin motion

$$\begin{aligned} &\dot{s}_n = \varepsilon_{nmk}\Omega_m s_k = \{H_1, H_2, s_n\},\ H_1 = \Omega\cdot s,\ H_2 = s^2, \\ &\{A, B, C\} = \varepsilon_{nmk}\partial_n A\partial_m B\partial_k C, \end{aligned} \tag{36}$$

$$\Omega_n = \frac{-e}{m\gamma c}((1+k\gamma)B_n - k\frac{(B\cdot\wp)\wp_n}{m^2c^2(1+\gamma)} + \frac{1+k(1+\gamma)}{mc(1+\gamma)}\varepsilon_{nmk}E_m\wp_k) \tag{37}$$

where, parameters e and m are the charge and the rest mass of the particle, c is the velocity of light, $k=(g-2)/2$ quantifies the anomalous spin g factor, γ is the Lorentz factor, p_n are components of the kinetic momentum vector, E_n and B_n are the electric and magnetic fields, and A_n and Φ are the vector and scalar potentials;

$$B_n = \varepsilon_{nmk}\partial_m A_k,\ \ E_n = -\partial_n\Phi - \frac{1}{c}\dot{A}_n,$$
$$\gamma = \frac{H-e\Phi}{mc^2} = \sqrt{1+\frac{\wp^2}{m^2c^2}} \tag{38}$$

The spin motion equations we put in the Nambu-Poisson form. Hamiltonization of this dynamical system according to the general approach of the previous sections we will put in the ground of the optimal control theory of the accelerator.

The general method of Hamiltonization of the dynamical systems we can use also in the spinning particle case. Let us invent unified configuration space $q = (x, p, s),\ x_n = q_n,\ p_n = q_{n+3},\ s_n = q_{n+6},\ n = 1, 2, 3$; extended phase space, (q_n, ψ_n) and hamiltonian

$$H = H(q, \psi) = v_n \psi_n,\ n = 1, 2, ...9; \tag{39}$$

motion equations

$$\begin{aligned} \dot{q}_n &= v_n(q), \\ \dot{\psi}_n &= -\frac{\partial v_m}{\partial q_n} \psi_m \end{aligned} \tag{40}$$

where the velocities v_n depends on external fields as in previous section as control parameters which can be determined according to the optimal control criterium.

EDM are one of the keys to understand the origin of our Universe [Sakharov, 1967]. Andrei Sakharov formulated three conditions for baryogenesis:
1. Early in the evolution of the universe, the baryon number conservation must be violated sufficiently strongly,
2. The C and CP invariances, and T invariance thereof, must be violated, and
3. At the moment when the baryon number is generated, the evolution of the universe must be out of thermal equilibrium.
CP violation in kaon decays is known since 1964, it has been observed in B-decays and charmed meson decays. The Standard Model (SM) accommodates CP violation via the phase in the Cabibbo-Kobayashi-Maskawa matrix.
CP and P violation entail nonvanishing P and T violating electric dipole moments (EDM) of elementary particles $\vec{d} = d\vec{s}$.

Although extremely successful in many aspects, the SM has at least two weaknesses: neutrino oscillations do require extensions of the SM and, most importantly, the SM mechanisms fail miserably in the expected baryogenesis rate.
Simultaneously, the SM predicts an exceedingly small electric dipole moment of nucleons $10^{-33} < d_n < 10^{-31} e \cdot cm$, way below the current upper bound for the neutron EDM, $d_n < 2.9 \times 10^{-26} e \cdot cm$. In the quest for physics beyond the SM one could follow either the high energy trail or look into new methods which offer very high precision and sensitivity.
Supersymmetry is one of the most attractive extensions of the SM and S. Weinberg emphasized [Weinberg, 1993]: "Endemic in supersymmetric (SUSY) theories are CP violations that go beyond the SM. For this reason it may be that the next exciting thing to come along will be the discovery of a neutron electric dipole moment."

The SUSY predictions span typically $10^{-29} < d_n < 10^{-24} e \cdot cm$ and precisely this range is targeted in the new generation of EDM searches [Roberts, Marciano, 2010]. There is consensus among theorists that measuring the EDM of the proton, deuteron and helion is as important as that of the neutron. Furthermore, it has been argued that T-violating nuclear forces could substantially enhance nuclear EDM [Flambaum, Khriplovich, Sushkov, 1986]. At the moment, there are no significant direct upper bounds available on d_p or d_d. Non-vanishing EDMs give rise to the precession of the spin of a particle in an electric field. In the rest frame of a particle

$$\dot{s}_n = \varepsilon_{nmk}(\Omega_m s_k + d_m E_k),\ \Omega_m = -\mu B_m, \tag{41}$$

where in terms of the lab frame fields

$$\begin{aligned} B_n &= \gamma(B_n^l - \varepsilon_{nmk}\beta_m E_k^l), \\ E_n &= \gamma(E_n^l + \varepsilon_{nmk}\beta_m B_k^l) \end{aligned} \tag{42}$$

Now we can apply the Hamiltonization and optimal control theory methods to this dynamical system.

Note that the procedure of reduction of the higher order dynamical system, e.g. second order Euler-Lagrange motion equations, to the first order dynamical systems, in the case to the Hamiltonian motion equations, can be continued using fractal calculus. E.g. first order system can be reduced to the half order one,

$$\begin{aligned} &D^{1/2}q = \psi, \\ &D^{1/2}\psi = p \Leftrightarrow \dot{q} = p. \end{aligned} \tag{43}$$

Computers are physical devices and their behavior is determined by physical laws. The Quantum Computations [Benenti, Casati, Strini, 2004 , Nielsen, Chuang, 2000], Quantum Computing, Quanputing [Makhaldiani, 2007.2], is a new interdisciplinary field of research, which benefits from the contributions of physicists, computer scientists, mathematicians, chemists and engineers. Contemporary digital computer and its logical elements can be considered as a spatial type of discrete dynamical systems [Makhaldiani, 2001]

$$S_n(k+1) = \Phi_n(S(k)), \tag{44}$$

where

$$S_n(k), \quad 1 \leq n \leq N(k), \tag{45}$$

is the state vector of the system at the discrete time step k. Vector S may describe the state and Φ transition rule of some Cellular Automata [Toffoli, Margolus, 1987].The systems of the type (44) appears in applied mathematics as an explicit finite difference scheme approximation of the equations of the physics [Samarskii, Gulin, 1989].

Definition: *We assume that the system (44) is time-reversible if we can define the reverse dynamical system*

$$S_n(k) = \Phi_n^{-1}(S(k+1)). \tag{46}$$

In this case the following matrix

$$M_{nm} = \frac{\partial \Phi_n(S(k))}{\partial S_m(k)}, \tag{47}$$

is regular, i.e. has an inverse. If the matrix is not regular, this is the case, for example, when $N(k+1) \neq N(k)$, we have an irreversible dynamical system (usual digital computers and/or corresponding irreversible gates).

Let us consider an extension of the dynamical system (44) given by the following action function

$$A = \sum_{kn} l_n(k)(S_n(k+1) - \Phi_n(S(k))) \tag{48}$$

and corresponding motion equations

$$\begin{aligned} S_n(k+1) &= \Phi_n(S(k)) = \frac{\partial H}{\partial l_n(k)}, \\ l_n(k-1) &= l_m(k)\frac{\partial \Phi_m(S(k))}{\partial S_n(k)} = l_m(k)M_{mn}(S(k)) = \frac{\partial H}{\partial S_n(k)}, \end{aligned} \tag{49}$$

where

$$H = \sum_{kn} l_n(k)\Phi_n(S(k)), \tag{50}$$

is discrete Hamiltonian. In the regular case, we put the system (49) in an explicit form

$$\begin{aligned} S_n(k+1) &= \Phi_n(S(k)), \\ l_n(k+1) &= l_m(k)M^{-1}_{mn}(S(k+1)). \end{aligned} \tag{51}$$

From this system it is obvious that, when the initial value $l_n(k_0)$ is given, the evolution of the vector $l(k)$ is defined by evolution of the state vector $S(k)$. The equation of motion for $l_n(k)$ - Elenka is linear and has an important property that a linear superpositions of the solutions are also solutions.

Statement: *Any time-reversible dynamical system (e.g. a time-reversible computer) can be extended by corresponding linear dynamical system (quantum - like processor) which is controlled by the dynamical system and has a huge computational power,*

[Makhaldiani, 2001, Makhaldiani, 2002, Makhaldiani, 2007.2, Makhaldiani, 2011.2].

For motion equations (49) in the continual approximation, we have

$$
\begin{aligned}
&S_n(k+1) = x_n(t_k+\tau) = x_n(t_k) + \dot{x}_n(t_k)\tau + O(\tau^2),\\
&\dot{x}_n(t_k) = v_n(x(t_k)) + O(\tau), \quad t_k = k\tau,\\
&v_n(x(t_k)) = (\Phi_n(x(t_k)) - x_n(t_k))/\tau;\\
&M_{mn}(x(t_k)) = \delta_{mn} + \tau \frac{\partial v_m(x(t_k))}{\partial x_n(t_k)}.
\end{aligned} \tag{52}
$$

(de)Coherence criterion: *the system is reversible, the linear (quantum, coherent, soul) subsystem exists, when the matrix M is regular,*

$$
detM = 1 + \tau \sum_n \frac{\partial v_n}{\partial x_n} + O(\tau^2) \neq 0. \tag{53}
$$

For the Nambu - Poisson dynamical systems (see e.g. [Makhaldiani, 2007])

$$
\begin{aligned}
&v_n(x) = \varepsilon_{nm_1m_2...m_p} \frac{\partial H_1}{\partial x_{m_1}} \frac{\partial H_2}{\partial x_{m_2}} ... \frac{\partial H_p}{\partial x_{m_p}}, \quad p = 1, 2, 3, ..., N-1,\\
&\sum_n \frac{\partial v_n}{\partial x_n} \equiv divv = 0.
\end{aligned} \tag{54}
$$

Let me motivate an idea of construction of the reversible dynamical systems by simple example from field theory. There are renormalizable models of scalar field theory of the form (see, e.g. [Makhaldiani, 1980])

$$L = \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) - g\varphi^n, \tag{55}$$

with the constraint

$$n = \frac{2d}{d-2} \ , \tag{56}$$

where d is dimension of the space-time and n is degree of nonlinearity. It is interesting that if we define d as a function of n, we find

$$d = \frac{2n}{n-2} \tag{57}$$

the same function !
Thing is that, the constraint can be put in the symmetric implicit form [Makhaldiani, 1980]

$$\frac{1}{n} + \frac{1}{d} = \frac{1}{2} \tag{58}$$

Generalization of the idea

Now it is natural to consider the following symmetric function

$$f(y) + f(x) = c \tag{59}$$

and define its solution

$$y = f^{-1}(c - f(x)). \tag{60}$$

This is the general method, that we will use in the following construction of the reversible dynamical systems. In the simplest case,

$$f(x) = x, \tag{61}$$

we take

$$y = S(k+1), \quad x = S(k-1), \quad c = \tilde{\Phi}(S(k)) \tag{62}$$

and define our reversible dynamical system from the following symmetric, implicit form (see also [Toffoli, Margolus, 1987])

$$S(k+1) + S(k-1) = \tilde{\Phi}(S(k)), \tag{63}$$

explicit form of which is

$$\begin{aligned} S(k+1) &= \Phi(S(k), S(k-1)) \\ &= \tilde{\Phi}(S(k)) - S(k-1). \end{aligned} \tag{64}$$

Generalization of the idea

This dynamical system defines given state vector by previous two state vectors. We have reversible dynamical system on the time lattice with time steps of two units,

$$\begin{aligned} &S(k+2,2) = \Phi(S(k,2)), \\ &S(k+2,2) \equiv (S(k+2), S(k+1)), \\ &S(k,2) \equiv (S(k), S(k-1))). \end{aligned} \tag{65}$$

Internal, spin, degrees of freedom

Starting from a general discrete dynamical system, we obtained reversible dynamical system with internal(spin,bit) degrees of freedom

$$\begin{aligned} S_{ns}(k+2) &\equiv \begin{pmatrix} S_n(k+2) \\ S_n(k+1) \end{pmatrix} = \begin{pmatrix} \Phi_n(\Phi(S(k)) - S(k-1)) - S(k)) \\ \Phi_n(S(k)) - S_n(k-1) \end{pmatrix} \\ &\equiv \Phi_{ns}(S(k)), \quad s = 1, 2 \end{aligned} \tag{66}$$

where

$$S(k) \equiv (S_{ns}(k)), \quad S_{n1}(k) \equiv S_n(k), \quad S_{n2}(k) \equiv S_n(k-1) \tag{67}$$

For the extended system we have the following action

$$A = \sum_{kns} l_{ns}(k)(S_{ns}(k+2) - \Phi_{ns}(S(k))) \tag{68}$$

and corresponding motion equations

$$\begin{aligned} S_{ns}(k+2) &= \Phi_{ns}(S(k)) = \frac{\partial H}{\partial l_{ns}(k)}, \\ l_{ns}(k-2) &= l_{mt}(k)\frac{\partial \Phi_{mt}(S(k))}{\partial S_{ns}(k)} \\ &= l_{mt}(k) M_{mtns}(S(k)) = \frac{\partial H}{\partial S_{ns}(k)}, \end{aligned} \tag{69}$$

By construction, we have the following reversible dynamical system

$$\begin{aligned} S_{ns}(k+2) &= \Phi_{ns}(S(k)), \\ l_{ns}(k+2) &= l_{mt}(k) M^{-1}_{mtns}(S(k+2)), \end{aligned} \tag{70}$$

with classical S_{ns} and quantum l_{ns}(in the external, background S) string bit dynamics.

p-point cluster and higher spin states reversible dynamics, or pit string dynamics

We can also consider p-point generalization of the previous structure,

$$\begin{aligned}
&f_p(S(k+p)) + f_{p-1}(S(k+p-1)) + ... + f_1(S(k+1)) \\
&+f_1(S(k-1)) + ... + f_p(S(k-p)) = \tilde{\Phi}(S(k)), \\
&S(k+p) = \Phi(S(k), S(k+p-1), ..., S(k-p)) \\
&\equiv f_p^{-1}(\tilde{\Phi}(S(k)) - f_{p-1}(S(k+p-1)) - ... - f_p(S(k-p)))
\end{aligned} \tag{71}$$

and corresponding reversible p-oint cluster dynamical system

$$\begin{aligned}
&S(k+p,p) \equiv \Phi(S(k,p)), \\
&S(k+p,p) \equiv (S(k+p), S(k+p-1), ..., S(k+1)), \\
&S(k,p) \equiv (S(k), S(k-1), ..., S(k-p+1)), \quad S(k,1) = S(k).
\end{aligned} \tag{72}$$

So we have general method of construction of the reversible dynamical systems on the time (tame) scale p. The method of linear extension of the reversible dynamical systems (see [Makhaldiani, 2001] and previous section) defines corresponding Quanputers,

$$\begin{aligned}
&S_{ns}(k+p) = \Phi_{ns}(S(k)), \\
&l_{ns}(k+p) = l_{mt}(k) M^{-1}_{mtns}(S(k+p)),
\end{aligned} \tag{73}$$

p-point cluster and higher spin states reversible dynamics, or pit string dynamics

This case the quantum state function $l_{ns},\ s = 1, 2, ...p$ will describes the state with spin $(p-1)/2$.

Note that, in this formalism for reversible dynamics minimal value of the spin is $1/2$. There is not a place for a scalar dynamics, or the scalar dynamics is not reversible. In the Standard model (SM) of particle physics, [Beringer et al, 2012], all of the fundamental particles, leptons, quarks and gauge bosons have spin. Only scalar particles of the SM are the Higgs bosons. Perhaps the scalar particles are composed systems or quasiparticles like phonon, or Higgs dynamics is not reversible (a mechanism for 'time arrow').

Social profit of big collaborations

Nowadays there are several big collaborations in science, e.g. LHC. Scientific value of LHC depends on three components, the highest quality of accelerator, highest quality of detectors and distributed data processing. The first two components need good mathematical and physical modeling. Third component and the collaboration as a social structure are not under (anther) the control by scientific methods and corresponding modeling. By definition, scientific collaborations (SC) have a main scientific aim: to obtain answer on the important scientific question(s) and maybe gain extra scientific bonus: new important questions and discoveries. SC is more open information system than e.g. finance or military systems. So, it is possible to describe and optimize SC by scientific methods. Profit from scientific modeling of SC maybe also for other information systems and social structures.

As an example of GRID we take LHC Computing Grid. The LHC Computing Grid (LCG), is an international collaborative project that consists of a grid-based computer network infrastructure incorporating over 170 computing centers in 36 countries. It was designed by CERN to handle the prodigious volume of data produced by Large Hadron Collider (LHC) experiments. The Large Hadron Collider at CERN was designed to prove or disprove the existence of the Higgs boson, an important but elusive piece of knowledge that had been sought by particle physicists for over 40 years. A very powerful particle accelerator was needed, because Higgs bosons might not be seen in lower energy experiments, and because vast numbers of collisions would need to be studied. Such a collider would also produce unprecedented quantities of collision data requiring analysis. Therefore, advanced computing facilities were needed to process the data. A design report was published in 2005. It was announced to be ready for data on 3 October 2008. It incorporates both private fiber optic cable links and existing high-speed portions of the public Internet. At the end of 2010, the Grid consisted of some 200,000 processing cores and 150 petabytes of disk space, distributed across 34 countries.

GRID and Quanputing

The data stream from the detectors provides approximately 300 GByte/s of data, which after filtering for "interesting events", results in a data stream of about 300 MByte/s. The CERN computer center, considered "Tier 0" of the LHC Computing Grid, has a dedicated 10 Gbit/s connection to the counting room. The project was expected to generate 27 TB of raw data per day, plus 10 TB of "event summary data", which represents the output of calculations done by the CPU farm at the CERN data center. This data is sent out from CERN to eleven Tier 1 academic institutions in Europe, Asia, and North America, via dedicated 10 Gbit/s links. This is called the LHC Optical Private Network. More than 150 Tier 2 institutions are connected to the Tier 1 institutions by general-purpose national research and education networks. The data produced by the LHC on all of its distributed computing grid is expected to add up to 10-15 PB of data each year. In total, the four main detectors at the LHC produced 13 petabytes of data in 2010. The Tier 1 institutions receive specific subsets of the raw data, for which they serve as a backup repository for CERN. They also perform reprocessing when recalibration is necessary. In 2015, CERN switched away from Scientific Linux to CentOS. Distributed computing resources for analysis by end-user physicists are provided by the Open Science Grid, Enabling Grids for E-sciencE, and LHC@home projects, http://wlcg.web.cern.ch/. Update of the Computing Models of the WLCG and the LHC Experiments:
http://cds.cern.ch/record/1695401/files/LCG-TDR-002.pdf

The idea of computations on quanputers is in finding of the needed (value of the) state (wave function $\psi(t,x)$) from the initial, easy constructible, state ($\psi(0,x)$,) which is superposition of different states, including interesting one, with the same weight. During the computation the weight of the interesting state is growing till the value when we can guess the solution of the problem and then test it, which is much more easier then to find it.
Let us consider the following nonlinear evolution equation

$$iV_t = \Delta V - \frac{1}{2}V^2 + J, \tag{74}$$

extended Lagrangian and Hamiltonian

$$\begin{aligned} L &= \int dx^D (iV_t - \Delta V + \frac{1}{2}V^2 - J)\psi, \\ H &= \int dx^D (\Delta V - \frac{1}{2}V^2 + J)\psi \end{aligned} \tag{75}$$

and corresponding Hamiltonian motion equations

$$iV_t = \Delta V - \frac{1}{2}V^2 + J = \{V, H\},$$
$$i\psi_t = -\Delta\psi + V\psi = \{\psi, H\},$$
$$\{V(t,x), \psi(t,y)\} = \delta^D(x-y) \tag{76}$$

The solution of the problem is given in the form

$$|T) = U(T)|0), \ \psi(t,x) =< x|t), \ U(T) = Pexp(-i\int_0^T dtH(t)) \tag{77}$$

Under the programming of the quanputer we understand construction of the potential V, or the corresponding Hamiltonian. For the given potential, we calculate corresponding source J.
The discrete version of the system can be put in the form

$$S_m(n+1) = \Phi_n(S(n)) + J_m(n),$$
$$\Psi_m(n-1) = A_{mk}(S(n))\Psi_k(n), \ A_{mk}(S(n)) = \frac{\partial\Phi_k(S(n))}{\partial S_m(n)} \tag{78}$$

or, in the regular case, when the matrix A is regular, we obtain explicit form of the corresponding discrete dynamics

$$\begin{aligned} S_m(n+1) &= \Phi_n(S(n)) + J_m(n), \\ \Psi_m(n+1) &= A^{-1}_{mk}(S(n+1))\Psi_k(n), \end{aligned} \tag{79}$$

Now the state vector $S(n)$ and wave vector $\Psi_m(n)$ may correspond not only to the discrete values of the potential $V(n,m) = S_m(n)$, and wave function $\psi(n,m) = \Psi_m(n)$ but also any representation of the computing process from theoretical to experimental realization on a quanputer, including algorithm of solution, higher level programm realization of the algorithm. **Today, without big efforts, we can modify (some) GRID elements in time-invertible form. After development of the quanputer technologies, we can modify (some) GRID elements in quanputer forms.**

A way to the Solution of the Traveling salesman problem (TSP) with Quanputing

The $NP \stackrel{?}{=} P$ problem will be solved if for some $NP-$ complete problem, e.g. TSP, a polynomial algorithm find; or show that there is not such an algorithm; or show that it is impossible to find definite answer to that question.
TSP means to find minimal length path between N fixed points on a surface, which attends any point ones. We consider a system where N points with quenched positions $x_1, x_2, ..., x_N$ are independently distributed on a finite domain D with a probability density function $p(x)$. In general, the domain D is multidimensional and the points x_n are vectors in the corresponding Euclidean space. Inside the domain D we consider a polymer chain composed of N monomers whose positions are denoted by $y_1, y_2, ..., y_N$. Each monomer y_n is attached to one of the quenched sites x_m and only one monomer can be attached to each site. The state of the polymer is described by a permutation $\sigma \in \Sigma_N$ where Σ_N is the group of permutations of N objecs.

The Hamiltonian for the system is given by

$$H = \sum_{n=1}^{N} V(|y_n - y_{n-1}|) \tag{80}$$

Here V is the interaction between neighboring monomers on the polymer chain. For convenience the chain is taken to be closed, thus we take the periodic boundary condition $x_0 = x_N$. A physical realization of this system is one where the x_n are impurities where the monomers of a polymer loop are pinned. In combinatorial optimization, if one takes $V(x)$ to be the norm, or distance, of the vector x then $H(\sigma)$ is the total distance covered by a path which visits each site x_n exactly once. The problem of finding σ_0 which minimizes $H(\sigma)$ is known as the traveling salesman problem (TSP) [Gutin, Pannen, 2002].

In field theory language to the TSP we correspond the calculation of the following correlator

$$
\begin{aligned}
&G_{2N}(x_1,x_2,...,x_N) = Z_0^{-1}\int d\varphi(x)\varphi^2(x_1)\varphi^2(x_2)...\varphi^2(x_N)e^{-S(\varphi)}\\
&= \frac{\delta^{2N}F(J)}{\delta J(x_1)^2...\delta J(x_N)^2},\ F(J) = \ln Z(J),\\
&Z(J) = \int d\varphi e^{-\frac{1}{2}\varphi\cdot A\cdot\varphi + J\cdot\varphi} = e^{\frac{1}{2}J\cdot A^{-1}\cdot J},\ A^{-1}(x,y;m) = e^{-m|x-y|},\\
&L_{min}(x_1,...,x_N) = -\frac{d}{dm}\ln G_{2Ns} + O(e^{-am})\\
&< A^{-1} > \equiv \frac{1}{\Gamma(s)}\int_0^\infty dm m^{s-1}A^{-1}(x,y;m) = \frac{1}{|x-y|^s}\\
&= L_s A^{-1}(x-y;s)\\
&k(d)\Delta_d L_s A^{-1}(x;s) = \delta^d(x) \Rightarrow A(x;s) = k(d)\Delta_d L_s,\\
&s = d-2; \varphi = \varphi(x,m).
\end{aligned}
\tag{81}
$$

If we take relativistic massive scalar field, then $A = \Delta_d + m^2$,

$$A^{-1}(x) \sim |x|^{2-d} e^{-m|x|}, \tag{82}$$

and for $d = 2$, we also have the needed behaviour. Note that G_{2N} is symmetric with respect to its arguments and contains any paths including minimal length one.

Quantum field theory (QFT) and Fractal calculus (FC) provide Universal language of fundamental physics (see e.g. [Makhaldiani, 2011]). In QFT existence of a given theory means, that we can control its behavior at some scales (short or large distances) by renormalization theory [Collins, 1984]. If the theory exists, than we want to solve it, which means to determine what happens on other (large or short) scales. This is the problem (and content) of Renormdynamics. The result of the Renormdynamics, the solution of its discrete or continual motion equations, is the effective QFT on a given scale (different from the initial one).

Perturbation theory series (PTS) have the following qualitative form

$$f(g) = f_0 + f_1 g + ... + f_n g^n + ..., \ f_n = n! P(n)$$
$$f(x) = \sum_{n \geq 0} P(n) n! x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x}, \ \delta = x\frac{d}{dx} \quad (83)$$

So, we reduce previous series to the standard geometric progression series. This series is convergent for $|x| < 1$ or for $|x|_p = p^{-k} < 1, \ x = p^k a/b, \ k \geq 1$. With proper nomalization of the expansion parametre, the coefficients of the series are rational numbers and if experimental data indicates for some prime value for g, e.g. in QED

$$g = \frac{e^2}{4\pi} = \frac{1}{137.0...} \quad (84)$$

then we can take corresponding prime number and consider p-adic convergence of the series. In the case of QED, we have

$$f(g) = \sum f_n p^{-n}, \ f_n = n! P(n), \ p = 137, \ |f|_p \leq \sum |f_n|_p p^n \quad (85)$$

In the Yukawa theory of strong interactions (see e.g. [Bogoliubov, Shirkov, 1959]), we take $g = 13$,

$$f(g) = \sum f_n p^n, \; f_n = n!P(n), \; p = 13,$$
$$|f|_p \leq \sum |f_n|_p p^{-n} < \frac{1}{1-p^{-1}} \tag{86}$$

So, the series is convergent. If the limit is rational number, we consider it as an observable value of the corresponding physical quantity.
In $MSSM$ (see [Kazakov, 2004]) coupling constants unifies at $\alpha_u^{-1} = 26.3 \pm 1.9 \pm 1$. So,

$$23.4 < \alpha_u^{-1} < 29.2 \tag{87}$$

Question: how many primes are in this interval?

$$24, \; 25, \; 26, \; 27, \; 28, \; 29 \tag{88}$$

Only one!
Proposal: take the value $\alpha_u^{-1} = 29.0...$ which will be two orders of magnitude more precise prediction and find the consequences for the SM scale observables.

Let us make more explicit the formal representation of (83)

$$f(x) = \sum_{n\geq 0} P(n) n! x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x},$$
$$= P(\delta)\int_0^\infty dt e^{-t} t^\delta \frac{1}{1-x} = P(\delta)\int_0^\infty dt \frac{e^{-t}}{1+(-x)t},\ \delta = x\frac{d}{dx} \quad (89)$$

This integral is well defined for negative values of x. The Mathematica answer for the corresponding integral is

$$I(x) = \int_0^\infty dt \frac{e^{-t}}{1+xt} = e^{1/x}\Gamma(0, 1/x)/x,\ Im(x)\neq 0,\ Re(x)\geq 0, \quad (90)$$

where $\Gamma(a,z)$ is the incomplete gamma function

$$\Gamma(a,z) = \int_z^\infty dt t^{a-1} e^{-t} \quad (91)$$

For $x = 0.001,\ I(x) = 0.999$

The Goldberger-Treiman relation (GTR) [Goldberger, Treiman, 1958] plays an important role in theoretical hadronic and nuclear physics. GTR relates the Meson-Nucleon coupling constants to the axial-vector coupling constant in β-decay:

$$g_{\pi N} f_{\pi} = g_A m_N \tag{92}$$

where m_N is the nucleon mass, g_A is the axial-vector coupling constant in nucleon β-decay at vanishing momentum transfer, f_π is the π decay constant and $g_{\pi N}$ is the $\pi - N$ coupling constant.
Since the days when the Goldberger-Treiman relation was discovered, the value of g_A has increased considerably. Also, f_π decreased a little, on account of radiative corrections. The main source of uncertainty is $g_{\pi N}$.

If we take

$$\alpha_{\pi N} = \frac{g_{\pi N}^2}{4\pi} = 13 \Rightarrow g_{\pi N} = 12.78 \tag{93}$$

the proton mass $m_p = 938MeV$ and $f_\pi = 93MeV$, from (92), we find

$$g_A = \frac{f_\pi g_{\pi N}}{m_N} = \frac{93 \times \sqrt{52\pi}}{938} = 1.2672 \tag{94}$$

which is in agreement with contemporary experimental value
$g_A = 1.2695(29)$
In an old version of the unified theory [Heisenberg 1966], for the $\alpha_{\pi N}$ the following value were found

$$\alpha_{\pi N} = 4\pi(1 - \frac{m_\pi^2}{3m_p^2}) = 12.5 \tag{95}$$

Determination of $g_{\pi N}$ from $NN, N\bar{N}$ and πN data by the Nijmegen group [Rentmeester et al, 1999] gave the following value

$$g_{\pi N} = 13.05 \pm .08,\ \Delta = 1 - \frac{g_A m_N}{g_{\pi N} f_\pi} = .014 \pm .009,$$
$$13.39 < \alpha_{\pi N} < 13.72 \tag{96}$$

This value is consistent with assumption $g_{\pi N} = 13 \Rightarrow \alpha_{\pi N} = 13.45$
Due to the smallness of the u and d quark masses, Δ is necessarily very small, and its determination requires a very precise knowledge of the $g_{\pi N}$ coupling (g_A and f_π are already known to enough precision, leaving most of the uncertainty in the determination of Δ to the uncertainty in $g_{\pi N}$).

QCD is the theory of the strong interactions with, as only inputs, one mass parameter for each quark species and the value of the QCD coupling constant at some energy or momentum scale in some renormalization scheme. This last free parameter of the theory can be fixed by Λ_{QCD}, the energy scale used as the typical boundary condition for the integration of the Renormdynamic (RD) equation for the strong coupling constant. This is the parameter which expresses the scale of strong interactions, the only parameter in the limit of massless quarks. While the evolution of the coupling with the momentum scale is determined by the quantum corrections induced by the renormalization of the bare coupling and can be computed in perturbation theory, the strength itself of the interaction, given at any scale by the value of the renormalized coupling at this scale, or equivalently by Λ_{QCD}, is one of the above mentioned parameters of the theory and has to be taken from experiment.

The RD equations play an important role in our understanding of Quantum Chromodynamics and the strong interactions. The beta function and the quarks mass anomalous dimension are among the most prominent objects for QCD RD equations. The calculation of the one-loop β-function in QCD has lead to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions
['t Hooft, 1972, Gross, Wilczek, 1973, Politzer, 1973].
The MS-scheme ['t Hooft, 1973] belongs to the class of massless schemes where the β-function does not depend on masses of the theory and the first two coefficients of the β-function are scheme-independent.

The Lagrangian of QCD with massive quarks in the covariant gauge is

$$\begin{aligned} L &= -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{q}_n(i\gamma D - m_n)q_n \\ &-\frac{1}{2\xi}(\partial A)^2 + \partial^\mu \bar{c}^a(\partial_\mu c^a + g f^{abc}A^b_\mu c^c) \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc}A^b_\mu A^c_\nu,\ (D_\mu)_{kl} = \delta_{kl}\partial_\mu - igt^a_{kl}A^a_\mu, \end{aligned} \tag{97}$$

$A^a_\mu, a = 1, ..., N_c^2 - 1$ are gluon; $q_n, n = 1, ..., n_f$ are quark; c^a are ghost fields; ξ is gauge parameter; t^a are generators of fundamental representation and f^{abc} are structure constants of the Lie algebra $[t^a, t^b] = if^{abc}t^c$, we consider an arbitrary compact semi-simple Lie group G. For QCD, $G = SU(N_c), N_c = 3$.
The RD equation for the coupling constant is

$$\begin{aligned} \dot{a} &= \beta(a) = \beta_2 a^2 + \beta_3 a^3 + \beta_4 a^4 + \beta_5 a^5 + O(a^6), \\ a &= \frac{\alpha_s}{4\pi} = (\frac{g}{4\pi})^2,\ \int_{a_0}^{a}\frac{da}{\beta(a)} = t - t_0 = \ln\frac{\mu^2}{\mu_0^2}, \end{aligned} \tag{98}$$

μ is the 't Hooft unit of mass, the renormalization point in the MS-scheme.

To calculate the β-function we need to calculate the renormalization constant Z of the coupling constant, $a_b = Za$, where a_b is the bare (unrenormalized) charge. The expression of the β-function can be obtained in the following way

$$0 = d(a_b\mu^{2\varepsilon})/dt = \mu^{2\varepsilon}(\varepsilon Za + \frac{\partial(Za)}{\partial a}\frac{da}{dt})$$
$$\Rightarrow \frac{da}{dt} = \beta(a,\varepsilon) = \frac{-\varepsilon Za}{\frac{\partial(Za)}{\partial a}} = -\varepsilon a + \beta(a),\ \beta(a) = a\frac{d}{da}(aZ_1) \quad (99)$$

where

$$\beta(a,\varepsilon) = \frac{D-4}{2}a + \beta(a) \quad (100)$$

is $D-$dimensional $\beta-$function and Z_1 is the residue of the first pole in ε expansion

$$Z(a,\varepsilon) = 1 + Z_1\varepsilon^{-1} + ... + Z_n\varepsilon^{-n} + ... \quad (101)$$

Since Z does not depend explicitly on μ, the β-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ.

Note that, presentation of Z in the form of expansion (101) is formal. If we take $\varepsilon = 1/p$ we can give the expansion p-adic sense. So, we will have renormalization factors Z as analytic functions of p-adic argument. Another possibility is to consider Z as an object of nonstandard analysis.
Then, for $D = 4 + 2\varepsilon,\ \varepsilon > 0$, we have UV fixed point

$$a_{UV} = \frac{\varepsilon}{b_2} \tag{102}$$

with almost UV asymptotic freedom for small a_0, small $\varepsilon > 0,\ D \gtrsim 4$, and true infrared asymptotic freedom, $a_{IR} = 0$, for $a_0 < a_{UV}$. For $a_0 > a_{UV}$ we have "usual" confinement.

For quark anomalous dimension, RD equation is

$$\dot{b} = \gamma(a) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \gamma_4 a^4 + O(a^5),$$
$$b(t) = b_0 + \int_{t_0}^{t} dt\gamma(a(t)) = b_0 + \int_{a_0}^{a} da\gamma(a)/\beta(a). \qquad (103)$$

To calculate the quark mass anomalous dimension $\gamma(g)$ we need to calculate the renormalization constant Z_m of the quark mass $m_b = Z_m m,\ m_b$ is the bare (unrenormalized) quark mass. Than we find the function $\gamma(g)$ in the following way

$$0 = \dot{m}_b = \dot{Z}_m m + Z_m \dot{m} = Z_m m((\ln Z_m)^{\cdot} + (\ln m)^{\cdot})$$
$$\Rightarrow \gamma(a) = -\frac{d\ln Z_m}{dt} = \dot{b} = -\frac{d\ln Z_m}{da}\frac{da}{dt} = -\frac{d\ln Z_m}{da}(-\varepsilon a + \beta(a))$$
$$= a\frac{dZ_{m1}}{da},\ b = -\ln Z_m = \ln\frac{m}{m_b}, \qquad (104)$$

where RD equation in $D-$dimension is

$$\dot{a} = -\varepsilon a + \beta(a) = \beta_1 a + \beta_2 a^2 + ... \tag{105}$$

and Z_{m1} is the coefficient of the first pole in the $\varepsilon-$expantion of the Z_m in MS-scheme

$$Z_m(\varepsilon, g) = 1 + Z_{m1}(g)\varepsilon^{-1} + Z_{m2}(g)\varepsilon^{-2} + ... \tag{106}$$

Since Z_m does not depend explicitly on μ and m, the γ_m-function is the same in all MS-like schemes.

Note that, in the case of QED, we have the IR fixed point

$$a_{IR} = \frac{\varepsilon}{b_2}, \tag{107}$$

for $D = 4 - 2\varepsilon$, and small a_0. For $a_0 < a_{IR}$ we have UV asymptotic freedom. This IR fixed point indicates deviation from the usual Coulomb's law and may have important consequences for astrophysical plasma dynamics.

Light millicharged particles with the mass from a few keV to several MeV, with charge $q = \epsilon e,\ \epsilon \sim 10^{-5} - 10^{-4}$ where considered in [Dolgov, 2014]: An existence of the millicharged particles with mass in keV - MeV range allows to: Explain the origin of galactic and intergalactic magnetic fields; Introduce DM with time dependent interaction with normal matter; To solve or smooth down the problems of galactic satellites, angular momentum, and cusps in galactic centres inherent to ΛCDM-cosmology; The model can be tested in direct experiment.
To these values of millicharge correspond the following deviation from 4-dimensionality

$$D = 4 - 2\varepsilon,\ \varepsilon = \frac{b_2\alpha}{4\pi}\epsilon^2 \sim 10^{-11} \tag{108}$$

LMP may contribute (dominates) in the mechanism of acceleration of the universe (if one sign of charges dominates).

RD equation,

$$\dot{a} = \beta_1 a + \beta_2 a^2 + ... \tag{109}$$

can be reparametrized,

$$a(t) = f(A(t)) = A + f_2 A^2 + ... + f_n A^n + ... = \sum_{n \geq 1} f_n A^n, \tag{110}$$

$$\begin{aligned}
&\dot{A} = b_1 A + b_2 A^2 + ... = \sum_{n \geq 1} b_n A^n, \\
&\dot{a} = \dot{A} f'(A) = (b_1 A + b_2 A^2 + ...)(1 + 2 f_2 A + ... + n f_n A^{n-1} + ...) \\
&= \beta_1 (A + f_2 A^2 + ... + f_n A^n + ...) + \beta_2 (A^2 + 2 f_2 A^3 + ...) + ... \\
&+ \beta_n (A^n + n f_2 A^{n+1} + ...) + ... \\
&= \beta_1 A + (\beta_2 + \beta_1 f_2) A^2 + (\beta_3 + 2\beta_2 f_2 + \beta_1 f_3) A^3 + \\
&... + (\beta_n + (n-1)\beta_{n-1} f_2 + ... + \beta_1 f_n) A^n + ... \\
&= \sum_{n, n_1, n_2 \geq 1} A^n b_{n_1} n_2 f_{n_2} \delta_{n, n_1 + n_2 - 1}
\end{aligned} \tag{111}$$

$$= \sum_{n,m\geq 1; m_1,\ldots,m_k\geq 0} A^n \beta_m f_1^{m_1} \ldots f_k^{m_k} f(n,m,m_1,\ldots,m_k),$$
$$f(n,m,m_1,\ldots,m_k) = \frac{m!}{m_1!\ldots m_k!} \delta_{n,m_1+2m_2+\ldots+km_k} \delta_{m,m_1+m_2+\ldots+m_k},$$
$$b_1 = \beta_1,\ b_2 = \beta_2 + f_2\beta_1 - 2f_2 b_1 = \beta_2 - f_2\beta_1,$$
$$b_3 = \beta_3 + 2f_2\beta_2 + f_3\beta_1 - 2f_2b_2 - 3f_3b_1 = \beta_3 + 2(f_2^2 - f_3)\beta_1,$$
$$b_4 = \beta_4 + 3f_2\beta_3 + f_2^2\beta_2 + 2f_3\beta_2 - 3f_4b_1 - 3f_3b_2 - 2f_2b_3, \ldots$$
$$b_n = \beta_n + \ldots + \beta_1 f_n - 2f_2 b_{n-1} - \ldots - nf_n b_1, \ldots \tag{112}$$

so, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but β_1.

We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{\beta_2}{\beta_1},\ f_3 = \frac{\beta_3}{2\beta_1} + f_2^2,\ \dots, f_n = \frac{\beta_n + \dots}{(n-1)\beta_1}, \dots \tag{113}$$

In the critical dimension of space-time, $\beta_1 = 0$, and we can change by reparametrization any coefficient but β_2 and β_3.
From the relations (112), in the critical dimenshion ($\beta_1 = 0$), we find that, we can define the minimal form of the RD equation

$$\dot{A} = \beta_2 A^2 + \beta_3 A^3, \tag{114}$$

We can solve (114) as implicit function,

$$u^{\beta_3/\beta_2} e^{-u} = c e^{\beta_2 t},\ u = \frac{1}{A} + \frac{\beta_3}{\beta_2} \tag{115}$$

then, as in the noncritical case, explicit solution will be given by reparametrization representation (110) [Makhaldiani, 2013].

If we know somehow the coefficients β_n, e.g. for first several exact and for others asymptotic values (see e.g. [Kazakov, Shirkov, 1980]) than we can construct reparametrization function (110) and find the dynamics of the running coupling constant. This is similar to the action-angular canonical transformation of the analytic mechanics (see e.g. [Faddeev, Takhtajan, 1990]).
Statement: The reparametrization series for a is p-adically convergent, when β_n and A are rational numbers.

Let us take the the anomalous dimension of some quantity

$$\gamma(a) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \ldots \tag{116}$$

and make reparametrization

$$a = f(A) = A + f_2 A^2 + f_3 A^3 + \ldots \tag{117}$$

$$\begin{aligned}&\gamma(a) = \gamma_1(A + f_2 A^2 + f_3 A^3 + \ldots) + \gamma_2(A^2 + 2f_2 A^3 + \ldots) + \gamma_3(A^3 + \ldots)\\ &= \Gamma_1 A + \Gamma_2 A^2 + \Gamma_3 A^3 + \ldots\\ &\Gamma_1 = \gamma_1,\ \Gamma_2 = \gamma_2 + \gamma_1 f_2,\ \Gamma_3 = \gamma_3 + 2\gamma_2 f_2 + \gamma_1 f_3, \ldots\end{aligned} \tag{118}$$

When $\gamma_1 \neq 0$, we can take $\Gamma_n = 0,\ n \geq 2$, if we define f_n as

$$f_2 = -\frac{\gamma_2}{\gamma_1},\ f_3 = -\frac{\gamma_3 + 2\gamma_2 f_2}{\gamma_1} = -\frac{\gamma_3 - 2\gamma_2^2/\gamma_1}{\gamma_1}, \ldots \tag{119}$$

So, we get the exact value for the anomalous dimension

$$\gamma(A) = \gamma_1 A = \gamma_1 f^{-1}(a) = \gamma_1(a + \gamma_2/\gamma_1 a^2 + \gamma_3/\gamma_1 a^3 + \ldots :) \tag{120}$$

While it has been well established in the perturbative regime at high energies, QCD still lacks a comprehensive solution at low and intermediate energies, even 40 years after its invention. In order to deal with the wealth of non-perturbative phenomena, various approaches are followed with limited validity and applicability. This is especially also true for lattice QCD, various functional methods, or chiral perturbation theory, to name only a few. In neither one of these approaches the full dynamical content of QCD can yet be included. Basically, the difficulties are associated with a relativistically covariant treatment of confinement and the spontaneous breaking of chiral symmetry, the latter being a well-established property of QCD at low and intermediate energies. As a result, most hadron reactions, like resonance excitations, strong and electroweak decays etc., are nowadays only amenable to models of QCD. Most famous is the constituent-quark model (CQM, 1964), which essentially relies on a limited number of effective degrees of freedom with the aim of encoding the essential features of low- and intermediate-energy QCD.

The CQM has a long history, and it has made important contributions to the understanding of many hadron properties, think only of the fact that the systematization of hadrons in the standard particle-data base follows the valence-quark picture. Namely the Q dependence of the nucleon form factor corresponds to three-constituent picture of the nucleon and is well described by the simple equation [Brodsky, Farrar,1973], [Matveev, Muradyan,Tavkhelidze,1973]

$$F(Q^2) \sim (Q^2)^{-2} \tag{121}$$

It was noted [Voloshin, Ter-Martyrosian, 1984] that parton densities given by the following solution

$$\begin{aligned} M_2(Q^2) &= \frac{3}{25} + \frac{2}{3}\omega^{32/81} + \frac{16}{75}\omega^{50/81}, \\ \bar{M}_2(Q^2) &= M_2^s(Q^2) = \frac{3}{25} - \frac{1}{3}\omega^{32/81} + \frac{16}{75}\omega^{50/81}, \\ M_2^G(Q^2) &= \frac{16}{25}(1 - \omega^{50/81}), \\ \omega &= \frac{\alpha_s(Q^2)}{\alpha_s(m^2)},\ Q^2 \in (5, 20)GeV^2,\ b = 9,\ \alpha_s(Q^2) \simeq 0.2 \end{aligned} \tag{122}$$

of the Altarelli-Parisi equation

$$\dot{M} = AM,\ M^T = (M_2, \bar{M}_2, M_2^s, M_2^G),$$
$$M_2 = \int_0^1 dxx(u(x) + d(x)),\ \bar{M}_2 = \int_0^1 dxx(\bar{u}(x) + \bar{d}(x)),$$
$$M_2^s = \int_0^1 dxx(s(x) + \bar{s}(x)),\ M_2^G = \int_0^1 dxxG(x),\ \dot{M} = Q^2 \frac{dM}{dQ^2}$$
$$A = -a(Q^2)\begin{pmatrix} 32/9 & 0 & 0 & -2/3 \\ 0 & 32/9 & 0 & -2/3 \\ 0 & 0 & 32/9 & -2/3 \\ -32/9 & -32/9 & -32/9 & 2 \end{pmatrix},\ a = (\frac{g}{4\pi})^2 \quad (123)$$

with the following "valence quark" initial condition at a scale m

$$M_2(m^2) = 1,\ \bar{M}_2 = M_2^s = M_2^G(m^2) = 0,\ \alpha_s(m^2) = 2 \quad (124)$$

gives the experimental values

$$M_2 = 0.44,\ \bar{M}_2 = M_2^s = 0.04,\ M_2^G = 0.48 \quad (125)$$

The APE has the following form

$$\begin{aligned}
&\dot{x}_1 = a(kx_1 - bx_4),\\
&\dot{x}_2 = a(kx_2 - bx_4),\\
&\dot{x}_3 = a(kx_3 - bx_4),\\
&\dot{x}_4 = -ak(x_1 + x_2 + x_3) + acx_4,\\
&k = 32/9,\ b = 2/3,\ c = 2
\end{aligned} \tag{126}$$

One of the integral of motion of this system is

$$H = x_1 + x_2 + x_3 + x_4 = 1 \tag{127}$$

Indeed,

$$\dot{H} = a(c - 3b) = 0. \tag{128}$$

The physical meaning of the integral is the statement that the momentum of the nucleon is equal to the sum of the constituent quark and gluon momenta.

Now, the equation for $x_4 = x$, using the integral o motion, we reduce to

$$\dot{x} = a((k+c)x - k) \Rightarrow \int \frac{dx}{(k+c)x-k} = adt = -\frac{da}{\beta_1 a}$$
$$\Rightarrow x_4(Q^2) = \frac{k}{k+c} + (x_{40} - \frac{k}{k+c})\omega^{(k+c)/\beta_1},\ t = \ln\frac{Q^2}{M^2},\ \omega = \frac{\alpha_s(Q^2)}{\alpha_s(M^2)}$$
$$x_1 + x_2 + x_3 = 3\bar{x} = \frac{c}{k+c} + (3\bar{x}_0 - \frac{c}{a+c})\omega^{(k+c)/\beta_1},$$
$$3\bar{x}_0 + x_{40} = 1 \tag{129}$$

Then,

$$x_1 - x_2 = (x_{10} - x_{20})\omega^{k/\beta_1} \to 0,\ x_1 - x_3 = (x_{10} - x_{30})\omega^{k/\beta_1} \to 0,$$
$$x_1, x_2, x_3 \to \bar{x} = \frac{c}{3(k+c)} = \frac{3}{25},\ x_4 \to \frac{k}{k+c} = \frac{16}{25},\ Q^2 \gg M^2 \tag{130}$$

The solution of the system is

$$x_n = \bar{x} + c_n\omega^{k/\beta_1} + d_n\omega^{(k+c)/\beta_1},\ n = 1,2,3,$$
$$x_4(Q^2) = \frac{k}{k+c} + (x_{40} - \frac{k}{k+c})\omega^{(k+c)/\beta_1},$$
$$d_1 = d_2 = d_3 = d,\ c_1 + c_2 + c_3 = 0 \tag{131}$$

For the VQM,

$$
\begin{aligned}
&x_{10} = 1 \Rightarrow \bar{x} + c_1 + d = 1,\\
&x_{20} = x_{30} = 0 \Rightarrow c_2 = c_3 = c \Rightarrow x_2 = x_3,\\
&c = -c_1/2,\ d = 1 - \bar{x} - c_1,\ \bar{x} + c + d = 0,\\
&c_1 = \frac{2}{3},\ \bar{x} = \frac{3}{25},\ d = 1 - \frac{3}{25} - \frac{2}{3} = \frac{16}{75},\\
&\frac{k}{\beta_1} = \frac{32}{81},\ \frac{k+c}{\beta_1} = \frac{50}{81},\ \beta_1 = 9,\ x_{40} = 0
\end{aligned} \tag{132}
$$

So, for valence quark model (VQCD), $\alpha_s(m^2) = 2$. We have seen, that for $\pi\rho N$ model $\alpha_{\pi\rho N} = 3$, and for πN model $\alpha_{\pi N} = 13$. It is nice that $\alpha_s^2 + \alpha_{\pi\rho N}^2 = \alpha_{\pi N}$. This relation can be seen, e.g., by considering pion propagator in the low energy πN model and in superposition of higher energy VQCD and $\pi\rho N$ models.
Note that to $\alpha_s = 2$ corresponds

$$g = \sqrt{4\pi\alpha_s} = 5.013 = 5+ \tag{133}$$

Phenomenological approach to the nonrelativistic potential-model study of Υ and ψ spectra leads to a static Coulombic Power-law potential of the form

$$V(r) = a(r) r^{2-d(r)} \sim \begin{array}{c} 1/r,\ r \sim 0.1 fm \\ r,\ r \sim 1. fm \end{array} \tag{134}$$

E.g. in the case of the Υ and small r

$$V(r) = \frac{4}{3}\frac{\alpha_s}{r},\ \alpha_s = \frac{2\pi}{b \ln r\Lambda},\ b = 9. \tag{135}$$

This behavior corresponds not only to the running fine structure constant but also to the running space dimension. Confinement-the point-like hadrons on the scales higher than hadronic, corresponds to the zero dimensional space for hadron constituents.
RD equations of QCD beyond the critical dimention has explicit dependence on the space dimension. When the dimension becomes running we should consider two dimensional renormdinamics

$$\begin{array}{l} \dot{a_1} = \beta_1(a_1, a_2),\ a_1 = a, \\ \dot{a_2} = \beta_2(a_1, a_2),\ a_2 = d \end{array} \tag{136}$$

The AdS/CFT duality provides a gravity description in a (d + 1)-dimensional AdS space-time in terms of a flat d-dimensional conformally-invariant quantum field theory defined at the AdS asymptotic boundary [Maldacena, 1999],[Gubser,Klebanov,Polyakov, 1998],[Witten, 1998]. Thus, in principle, one can compute physical observables in a strongly coupled gauge theory in terms of a classical gravity theory. The β-function for the nonperturbative effective coupling obtained from the LF holographic mapping in a positive dilaton modified AdS background is [Brodsky, de Tèramond, Deur, 2010]

$$\beta(\alpha_{AdS}) = -\frac{Q^2}{4k^2}\alpha_{AdS}(Q^2) = \alpha_{AdS}(Q^2)\ln\frac{\alpha_{AdS}(Q^2)}{\alpha(0)} \leq 0 \quad (137)$$

where the physical QCD running coupling in its nonperturbative domain is

$$\alpha_{AdS}(Q^2) = \alpha(0)e^{-Q^2/4k^2} \quad (138)$$

This renormdynamics interpolates between UV fixed point $\alpha(\infty) = 0$ and IR fixed point $\alpha(0) = 2$.

For the QCD running coupling [Diakonov, 2003]

$$\alpha(q^2) = \frac{4\pi}{9\ln(\frac{q^2+m_g^2}{\Lambda^2})} \tag{139}$$

where $m_g = 0.88GeV,\ \Lambda = 0.28GeV$, the $\beta-$function of renormdynamics is

$$\begin{aligned} \beta(q^2) &= -\frac{\alpha^2}{k}(1 - c\exp(-\frac{k}{\alpha})), \\ k &= \frac{4\pi}{9} = 1.40,\ c = \frac{m_g^2}{\Lambda^2} = (3.143)^2 = 9.88 \end{aligned} \tag{140}$$

for nontrivial (IR) fixed point we have

$$\alpha_{IR} = \frac{k}{\ln c} = 0.61 \tag{141}$$

For $\alpha(0) = 2$, we predict the gluon mass as

$$m_g = \Lambda e^{\frac{k}{2\alpha(0)}} = 1.42\Lambda = m_N/3,\ \Lambda = 220MeV. \tag{142}$$

The ghost-gluon interaction in Landau gauge has been determined either from DSEs [Zwanziger, 2002],[Lerche,von Smekal, 2002], or the Exact Renormalization Group Equations (ERGEs) [Pawlowski et al, 2004],[Fischer,Gies, 2004] and yield an IR fixed point

$$\alpha(0) = \frac{2\pi}{3N_c}\frac{\Gamma(3-2k)\Gamma(3+k)\Gamma(1+k)}{\Gamma(2-k)^2\Gamma(2k)} = \frac{8.9115}{N_c} = 2.970,$$
$$N_c = 3,\ k = (93-\sqrt{1201})/98 = 0.5954 \tag{143}$$

Note that, from this formula for $k = 0.6036$ we have $\alpha(0) = 3$ and for $k = 0.36$ we have $\alpha(0) = 2$.

Scale and conformal symmetries and zeros of the renormdynamic β-functions

The main motion equation of the renormdynamics

$$\dot{a} = \beta_a \tag{144}$$

has fixed points a_c in the zeros of the $\beta_a = \beta(a_c) = 0$. At these points corresponding field theory is scale and conformal symmetric. By reparametrization $a = f(A)$, we can change the form of the motion equation and particulary we can take the minimal form of the β- functions depending only on the reparametrization invariant coefficients, e.g. for QCD in critical $d = 4$ dimensions

$$\dot{a} = \beta_2 a^2 + \beta_3 a^2, \tag{145}$$

This case, we have the trivial zero $a_c = 0$, corresponding to the scale and conformal symmetry of QCD at small scales (Higher energies). There are an opinion that at low energy we have another, the nontrivial fixed point. Personally my believe is that the fixed point is $\alpha_s(M) = 2$ at the valence quark scale $M \sim 300MeV$. But it is obvious that the minimal form of the QCD renormdynamics (145) has not the finite nontrivial fixed point! How I can talk about the fixed point?

Thing is that, the original (complete, physical, if you like) β- function and the minimal one are connected as

$$\beta_a = f'(A)\beta_A, \tag{146}$$

so, when the minimal β- function has not the nontrivial fixed point-zero, that fixed point is given by critical point of the reparametrization function, $f(A),\ f'(A_c) = 0$. Then, when the minimal β- function has not the nontrivial zero, but we know somehow the fixed point, we can consider by corresponding reparametrization a next to the minimal forms of the β-function which will have the nontrivial fixed point.
If we do not know the value of the nontrivial fixed point, we can find its approximation value from the zeros of the reparametrization function $f(A)$, which reduce known approximation value of the β- function to the minimal one. For monotonic function $a = f(A), f'(A) \neq 0$ and we can define another time-parameter

$$d\tau = dt/|f'(A(t))| \tag{147}$$

We consider the following polynomial equation

$$P_n(z) - tz^{n+1} = 0,\ z \in C,\ t \in (0, \infty) \tag{148}$$

For small times t all zeros but one of this polynomial are near the zeros of the polynomial $P_n(z)$. For large times all $n+1$ zeros are near the zeros of the equation

$$a_0 - tz^{n+1} = 0,\ z_k = \sqrt[n+1]{a_0/t}\exp(2\pi i\frac{k}{n+1}),\ k = 0, 1, ..., n \tag{149}$$

It is interesting to know how far from others zero approach with time to the other zeros and then all of them organized as sites of symmetric polygon on the circle with decreasing radius.

For given solutions (zeros) $z_n,\ 1 \le n \le N$, the polynomial (coefficients) can be found from the following linear system

$$a_1 z_n + ... + a_N z_n^N = b_n = tz_n^{N+1} - a_0,\ 1 \le n \le N \tag{150}$$

by Crammer's forms

$$a_k = \frac{detA_k}{detA},\ 1 \le n \le N, \tag{151}$$

in general (regular) position, when all zeros are different, are simple.

The matrix A_k is the matrix formed by replacing the k-th column of A by the column vector b. In special (singular) case, when some of the zeros are identical, the Crammer's forms dos not work, for $t = 0$, but in the case of nontrivial deformations, $t \neq 0$, the singular case become regular and we can make difference between regular and singular cases by different behavior with respect to the deformation parameter t. The extra root z_{N+1} is far from other roots, for small t,

$$z_{N+1} = \frac{a_N}{t} + ... \tag{152}$$

In regular case main roots are linear functions of t, for small t. Note that $a_n, \ 1 \leq n \leq N$ do not depend on t, are invariants-integrals of motion.

Having N integrals of motion $a_k = H_k,\ 1 \leq k \leq N$ we construct Nambu-Poisson dynamics for the roots

$$\begin{array}{l} \dot{x}_n = \{x_n, H_1, H_2, ..., H_N\},\ 1 \leq n \leq N \\ \{t, H_1, H_2, ..., H_N\} = 1, \\ H_1 = a_{N+1}(x_1 + x_2 + ... + x_{N+1}), ..., \\ H_{N+1} = a_{N+1} x_1 x_2 ... x_{N+1} \end{array} \tag{153}$$

If we take the quantum Heisenberg equation as

$$i\hbar \dot{A} = [A, H_1, ..., H_N], \tag{154}$$

correspondence between classical and quantum brackets will be

$$[A_1, ..., A_{N+1}] \doteq i\hbar\{A_1, ..., A_{N+1}\} \tag{155}$$

As an example, we conceder the simplest case

$$
\begin{aligned}
&a_0 + a_1 x - t x^2 = a_2(x - x_1)(x - x_2) = 0,\\
&x_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4 t a_0}}{2t},\\
&a_2 = -t,\ a_1 = -a_2(x_1 + x_2) = H_1,\ a_0 = a_2 x_1 x_2 = -H_2\\
&\dot{x}_1 = \{x_1, H_1, H_2\} = f_{120} t x_1 x_2 + f_{102} t (x_1 + x_2) x_1,\\
&= -f_{012} t x_1^2\\
&\dot{x}_2 = \{x_2, H_1, H_2\} = f_{210} t x_1 x_2 + f_{201} t (x_1 + x_2) x_2,\\
&= f_{012} t x_2^2
\end{aligned} \tag{156}
$$

Motion equations are

$$
\begin{aligned}
&\dot{x}_1 = -\frac{x_1^2}{t(x_1 - x_2)},\ \dot{x}_2 = \frac{x_2^2}{t(x_1 - x_2)},\\
&f_{012} = \frac{1}{t^2(x_1 - x_2)}
\end{aligned} \tag{157}
$$

At a root x_0 of multiplicity k we have

$$\frac{P_N^{(k)}(x_0)}{k!}(x-x_0)^k + ... = t x_0^{N+1},$$
$$x_n(t) = x_0 + c_{n,k} t^{1/k},\ 1 \leq k \leq N$$
$$c_{n,k} = (\frac{x_0^{N+1} k!}{P_N^{(k)}(x_0)})^{\frac{1}{k}} \exp(2\pi i \frac{n}{k}),\ 0 \leq n \leq k-1 \tag{158}$$

So we can define the multiplicity of the root k from the time dependent of the roots.

The renormdynamic properties of Quantum Chromodynamics were the reason of acceptance of this theory as the theory of strong interactions. The central role played by the QCD β-function, calculated at the one-['t Hooft, 1972],[Gross, Wilczek, 1973],[Politzer, 1973], two-[Caswell, 1974],[Jones,1974], [Egorian,Tarasov, 1979], three-[Tarasov,Vladimirov,Zharkov,1980],[Larin,Vermaseren,1993] and finally at the four-loop [van Ritbergen,Vermaseren,Larin,1997] level, cannot be overestimated in this respect.

The minimal form of the QCD renormdynamics (RD) is

$$\dot{x} = -b_2 x^2 - b_3 x^3,$$
$$b_2 = 11 - \frac{2}{3}n,\ b_3 = 2(51 - \frac{19}{3}n),\ x = \frac{\alpha_s}{4\pi} = (\frac{g}{4\pi})^2, \quad (159)$$

where n is the number of the light quarks,e.g. $n = 3$ for energy scales less then the mass of the $c-$quark, $m_c \simeq 1GeV$ but higher than the mass of $s-$ quark, $m_s \simeq 100MeV$.

The Bjorken sum rule [Bjorken,1966] has been of central importance for studying the spin structure of the nucleon. Accounting for finite Q^2 corrections to the sum rule, it reads:

$$\int_0^1 (g_{p1} - g_{n1})dx = \frac{g_A}{6}(1 - \frac{\alpha_s}{\pi} - 3.58(\frac{\alpha_s}{\pi})^2 - 20.21(\frac{\alpha_s}{\pi})^3 + ...) + \sum_{k\geq 1} \frac{\mu_k}{Q^{2k}} \tag{160}$$

where the μ_k are higher twist terms. We take the following valence quark parametrization of the α_s

$$\int_0^1 (g_{p1} - g_{n1})dx = \frac{g_A}{6}(1 - \frac{\alpha_V}{2}),\ \alpha_V = \frac{2\alpha_s}{\pi} + ... \tag{161}$$

The Bjorken sum rule is related to a more general sum rule, the generalized Gerasimov-Drell-Hearn (GDH) sum rule
[Gerasimov, 1965],[Drell, Hearn, 1966],
[Anselmino,Ioffe,Leader,1989],[Ji,Osborne,2001]:

$$\int_0^1 (g_{p1} - g_{n1})dx = \frac{Q^2}{(4\pi)^2\alpha}(GDH_p(Q^2) - GDH_n(Q^2)) \tag{162}$$

Since the generalized GDH sum is, in principle, calculable at any Q^2, it can be used to study the transition from the partonic to hadronic degrees of freedom of the strong force. The Bjorken sum is the flavor non-singlet part of the GDH sum. This leads to simplifications that may help in linking the (χPT) validity domain to the pQCD validity domain [Burkert,2001]. Hence the Bjorken sum appears as a key quantity to study the hadron-parton transition.

According to the LEP and Tevatron data, the standard model coupling constants at the Z-boson mass scale take the values (see, e.g. [Kazakov, 2004])

$$\begin{aligned} \alpha_1(m_Z) &= 0.017, \ \alpha_1(m_Z)^{-1} = 58.8 \\ \alpha_2(m_Z) &= 0.034, \ \alpha_2(m_Z)^{-1} = 29.4 \\ \alpha_3(m_Z) &= 0.118, \ \alpha_3(m_Z)^{-1} = 8.47 \end{aligned} \tag{163}$$

$m_Z = mZ = 91.1875GeV$

$$\begin{aligned} \alpha_1(m_Z)^{-1} &= 58.8 \\ \alpha_2(m_Z)^{-1} &= 29.587 \\ \alpha_3(m_Z) &= 0.1184 \end{aligned} \tag{164}$$

Our aim is to consider RD equation in critical dimension for weak interaction part of the SM ($\varepsilon_2 = 0$); RD equations for the electromagnetic and strong interaction parts beyond critical dimension ($\varepsilon_1, \varepsilon_3 \neq 0$); reach unification (equality) of the three couplings at the TeV scale in the point $\alpha_u^{-1} = 31.0$

The solution of the one loop RD equation beyond critical dimension

$$\dot{a} = -\varepsilon a + ka^2,$$
$$a = \frac{\alpha}{4\pi} = (\frac{g}{4\pi})^2,\ t = \ln\frac{Q^2}{m_Z^2}, \tag{165}$$

is

$$a_n(t)^{-1} = \frac{k_n}{\varepsilon} + c_n e^{\varepsilon_n t},\ n = 1, 3$$
$$c_n = a_n(m_Z)^{-1} - \frac{k_n}{\varepsilon_n},$$
$$k_n = (\frac{41}{10}, -7). \tag{166}$$

The solution of the RD equation in critical dimension

$$\dot{a}_2 = k_2 a_2^2,\ k_2 = -\frac{19}{6} \tag{167}$$

is

$$a_2^{-1}(t) = a_2^{-1}(m_Z) + k_2 t \tag{168}$$

From the last expression, having unification value, $\alpha_2^{-1}(t_u) = \alpha_u^{-1} = 31.0$ we define the unification scale

$$t_u = (a_2^{-1}(t_u) - a_2^{-1}(m_Z))/k_2$$
$$= 4\pi \times 1.6 \times \frac{6}{19} = 6.35,$$
$$Q_u = 23.9 m_Z = 2182 GeV,$$
$$m_Z = 91.2 GeV \qquad (169)$$

Solution of the RD equation beyond the critical dimension for electrodynamic constant,

$$\dot{a} = -\varepsilon a + b a^2, \ b = \frac{41}{10}, \qquad (170)$$

is

$$a^{-1}(t) = \frac{b}{\varepsilon} + (a^{-1}(m_Z) - \frac{b}{\varepsilon}) e^{\varepsilon t} \qquad (171)$$

The condition of the unification

$$(b\varepsilon^{-1} - a^{-1}(t_u)) = (b\varepsilon^{-1} - a^{-1}(m_Z))e^{\varepsilon t_u} \tag{172}$$

defines the value $\varepsilon_1 = -0.093$ Unification takes place in dimension $d = 4 - 2\varepsilon_1 = 4.186$
For the strong coupling constant beyond the critical dimension,

$$\dot{a} = -\varepsilon a - ba^2,\ b = 7, \tag{173}$$

the solution is

$$a^{-1}(t) = -\frac{b}{\varepsilon} + (\frac{b}{\varepsilon} + a^{-1}(m_Z))e^{t\varepsilon}, \tag{174}$$

the unification condition

$$(b\varepsilon^{-1} + a^{-1}(t_u)) = (b\varepsilon^{-1} + a^{-1}(m_Z))e^{\varepsilon t_u} \tag{175}$$

defines $\varepsilon = 0.168$
Unification takes place in the dimension $d = 4 - 2\varepsilon = 3.66$

Let us consider unification at the point $\alpha^{-1}(t_u) = 29.0$, the low energy unification,

$$t_{ul} = (\alpha_2^{-1}(t_{ul}) - a_2^{-1}(m_Z))/k_2$$
$$= -4\pi \times 0.4 \times \frac{6}{19} = -1.59,$$
$$Q_{ul} = 0.45 m_Z = 41.2 GeV \quad (176)$$

For electrodynamic case unification condition

$$\frac{41}{10} - 4\pi 29\varepsilon = (\frac{41}{10} - 4\pi 58.8\varepsilon)e^{-1.59\varepsilon}, \quad (177)$$

gives the values $\varepsilon_1 = 0.453$, $d_{el} = 3.09 = 2.09 + 1$ dimensional space-time. For strong coupling constant unification condition

$$7 + 4\pi\varepsilon \times 29 = (7 + 4\pi\varepsilon \times 8.47)e^{-1.59\varepsilon} \quad (178)$$

gives $\varepsilon_3 = -0.8121,\ d_{sl} = 5.624$

The low energy value of the QED $\alpha^{-1} = 137.036...$
Let us find the scale at which $\alpha^{-1} = 137$ if

$$\alpha^{-1}(m_Z) = \frac{5}{3\cos^2\theta_W}\alpha_1^{-1}(m_Z) = 128.978 \pm 0.027 \simeq 129,$$
$$\sin^2\theta_W = 0.23146 \pm 0.00017 \simeq 0.2315,$$
$$\alpha_1^{-1}(m_Z) = 58.8,$$
$$\alpha^{-1}(m_Z) = \frac{5}{3 \times 0.7685} \times 58.8 = 127.52 \simeq 128 \qquad (179)$$

Now take one loop RD evolution to the 137,

$$t_l = (a_1^{-1}(t_l) - a_1^{-1}(m_Z))/k_1$$
$$= -4\pi \times 8. \times \frac{10}{41} = -24.5,$$
$$Q_l \simeq 5 \times 10^{-6} m_Z \simeq 5 \times 10^{-4} m_p \simeq m_e \qquad (180)$$

In nonrelativistic approximation the force between two dyons with electric and magnetic charges $\mathfrak{g}_n = (e_n, g_n),\ n = 1, 2$ is

$$F = \frac{\mathfrak{g}_1\mathfrak{g}_2\mathbf{r} + \mathfrak{g}_1 \times \mathfrak{g}_2\mathbf{v} \times \mathbf{r}}{4\pi r^3} \tag{181}$$

where

$$\mathfrak{g}_1\mathfrak{g}_2 = e_1e_2 + g_1g_2,\ \mathfrak{g}_1 \times \mathfrak{g}_2 = e_1g_2 - e_2g_1,\ \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1,\ \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1,\ c = 1(.$$

Note that, this force depends on the invariant dual combinations of charges: the combination

$$(e_1 - ig_1)(e_2 + ig_2) = e_1e_2 + g_1g_2 + i(e_1g_2 - e_2g_1) \tag{183}$$

is invariant with respect to the continual global dual transformations

$$e^{i\alpha}(e + ig) = e' + ig' \tag{184}$$

From Dirac's quantization of charge

$$eg = 2\pi\hbar cn, n = \pm 1, \pm 2, ..., \tag{185}$$

we have

$$g = \frac{e}{2\alpha}\hbar cn \tag{186}$$

In the natural system of units, $c = \hbar = 1$, and $n = 1$, elementary magnetic charge has the value

$$\begin{aligned} &g = 68.5e, \\ &\alpha_g = \frac{g^2}{4\pi} = (\frac{1}{2\alpha})^2\alpha = \frac{1}{4\alpha} = \frac{137}{4} = 34.25 \end{aligned} \tag{187}$$

The mass of the monopole we can estimate if we suppose that the classical radius of the monopole is not more than electron's one

$$\begin{aligned} &m_e = \frac{\alpha}{er_e},\ m_g = \frac{\alpha_g}{er_g}, \\ &r_g \le r_e \Rightarrow m_g \ge \frac{\alpha_g}{\alpha_e}m_e = \frac{m_e}{4\alpha^2} = 4692.25m_e \\ &\simeq 2398MeV \simeq 2.4TeV \end{aligned} \tag{188}$$

So, the Two-TeV unification takes place at the monopole scale.

At the critical point we may have low energy unification of the two abelian couplings, weak-electromagnetic and strong-monopole couplings. According to the Dirac quantization rule, for the electron-e and monopole-g charges we have

$$eg = 2\pi n,\ n = \pm 1, \pm 2, ... \tag{189}$$

so, at the selfdual, critical, point, we have prediction:

$$\alpha_e = \alpha_g = \frac{n}{2},\ n = 1, 2, ... \tag{190}$$

The minimal-fundamental value of the unified coupling constant is $\alpha = 1/2$. Schwinger constructed a quantum field theory of magnetic and electric charges which is relativistically invariant in consequence of the charge quantization condition $eg/\hbar c = 4\pi n,\ n$ integer, [Schwinger, 1966]. This is more restrictive than Dirac's condition, which would also allow half-integral values.
Now the minimal value at the unification point is 1. The next value is 2. These two values of coupling constant are connected as UV and IR fixed points of one monotone RD interval.

In the relativistic string-gauge field duality [Maldacena, 1999] (see review [Aharony et al, 2000]), the string coupling constant g_s and the gauge field fine structure constant α_s are related: $g_s = \alpha_s$. The statement that the later is (prime) integer means (prime) integer quantization of the string coupling constant.

Instanton configuration has the following value of classical action

$$S[A] = \int d^4x \frac{1}{4}(G^a_{\mu\nu}) = \frac{8\pi^2}{g^2} = \frac{2\pi}{\alpha} = 2\pi p,\ \alpha = p^{-1} \tag{191}$$

So, in Minkowski space we have not destructive interference between instanton contributions

$$e^{iS} = 1, \tag{192}$$

when $\alpha = p^{-1}$. When $\alpha = p$, we need at least p instantons in a cluster-molecule, we have cumulative action of p instantons. So, on the valence quark scale, gluon fields are presented implicitly as instanton clusters. When $p = 2$, we have instanton dipoles.

Yang-Mills theory and QCD are well-defined theories for an arbitrary (semi-)simple Lie group as gauge group. One remarkable choice for the group is the exceptional Lie group G_2 instead of the physical group $SU(3)$. Since its center is trivial the Wilson confinement criterion is not fulfilled, even in the pure Yang-Mills case. The reason is that any static fundamental charge can be screened by three adjoint charges, i. e. gluons. The most remarkable difference compared to the $SU(3)$ or $SU(2)$ case is the topological charge of the G_2 instanton
[Ernst-Michael Ilgenfritz, Axel Maas, 2012]

$$Q = \frac{1}{64\pi^2} \int d^4x F^a_{\mu\nu} \epsilon_{\mu\nu\rho\sigma} F^a_{\rho\sigma} = 2 \tag{193}$$

which is twice as large as the one of the (embedded) $SU(2)$ instanton.

In the case of the $N = 4$ super Yang-Mills the moduli space is the upper half plane parametrized by

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} = \frac{1}{2\pi}(\theta + i\frac{2\pi}{\alpha}) = \frac{\theta}{2\pi} + \frac{i}{\alpha} \tag{194}$$

The instanton part of the partition function is given by

$$Z = \sum_{k\geq 0} z_k q^k \tag{195}$$

where instanton parameter q is given by

$$q = e^{2\pi\tau i} = e^{-\frac{8\pi^2}{g^2}+i\theta},\ \tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \tag{196}$$

The Lagrangian is

$$L = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \theta\frac{g^2}{8\pi^2}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \theta\frac{g^2}{8\pi^2}\mathbf{E}\cdot\mathbf{B},$$
$$\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^a_{\rho\sigma} \tag{197}$$

The θ-term is a total derivative and does not contribute to the classical equation of motion. It does, however, change the canonical momentum from $P^a_n = F^a_{0n}$ to

$$P^a_n = F^a_{0n} + \theta\frac{g^2}{8\pi^2}B^a_n,\ n = 1,2,3 \tag{198}$$

and after canonical quantization in Weyl gauge one finds the Hamiltonian H_θ

$$H_\theta = \int d^3x(\frac{1}{2}(P^a_n - \theta\frac{g^2}{8\pi^2}B^a_n)^2 + \frac{1}{2}(B^a_n)^2) \tag{199}$$

An important historical role in establishing QCD as the theory of the strong interactions was played by $U(1)_A$ anomaly.The description of radiative decays of the pseudoscalar mesons in the framework of a gauge theory requires the existence of the electromagnetic axial anomaly and determines the number of colours to be $N_c = 3$. The compatibility of the symmetries of QCD with the absence of a ninth light pseudoscalar meson, the so called $U(1)_A$ problem, in turn depends on the contribution of the colour gauge fields to the anomaly. The anomaly-mediated link between quark dynamics and gluon topology (the non-perturbative dynamics of topologically nontrivial gluon configurations) is the key to understanding a range of phenomena in polarised QCD phenomenology, most notably the 'proton spin' sum rule for the first moment of the structure function g_{1p}. A wide variety of phenomena in QCD, ranging from the low-energy dynamics of the pseudoscalar mesons to sum rules in polarised deep-inelastic scattering reveal subtle aspects of quantum field theory, in particular topological gluon dynamics, which go beyond simple current algebra or parton model interpretations.

The pseudoscalar pion-nucleon interaction model at low energy and external pion condensate field reduce to the model given by Hamiltonian

$$H = \frac{p^2}{2M} + \frac{g^2\pi^2}{2M} + \frac{g}{2M}(\sigma\nabla)(\tau\pi), \tag{200}$$

If condensate is electro-neutral, $\pi_a = \delta_{a3}\pi$, then the Hamiltonian have the following supersymmetric form

$$H = \frac{1}{2M}\{Q^+, Q^-\},\ Q^{\pm} = (\sigma p \pm ig\pi(x))\tau^{\pm} \tag{201}$$

Graphene is a one-atom-thick allotrope of carbon, with unusual two-dimensional Dirac-like electronic excitations. The Dirac electrons can be controlled by application of external electric and magnetic fields, or by altering sample geometry and/or topology. The Dirac electrons behave in unusual ways in tunneling, confinement, and the integer quantum Hall effect. The electronic properties of graphene stacks are discussed and vary with stacking order and number of layers. Edge (surface) states in graphene depend on the edge termination (zigzag or armchair) and affect the physical properties of nanoribbons. Different types of disorder modify the Dirac equation leading to unusual spectroscopic and transport properties. There are different effects of electron-electron and electron-phonon interactions in single layer and multilayer graphene [Castro Neto et al, 2009].
Carbon is the materia prima for life and the basis of all organic chemistry. Because of the flexibility of its bonding, carbon-based systems show an unlimited number of different structures with an equally large variety of physical properties.

The effective field model of graphene (EFG) monolayer without the Coulomb interactions is a good approximation to the original tight - binding model. The EFG model operates with the continuum Dirac field living in the graphene sheet.

The principle feature of graphene is that the quasi-particle excitations satisfy the Dirac equation, where the speed of light c is replaced by the so-called Fermi velocity $v_F \simeq c/300$. Therefore, the quantum field theory methods are very useful in the physics of graphene. By applying these methods, one can explain anomalous Hall Effect in graphene, the universal optical absorption rate, the Faraday effect, and predict the Casimir interaction of graphene, and do much more (see, e.g. [Fialkovsky, Vassilevich, 2011]).

The Dirac model for quasi-particles in graphene was elaborated in full around 1984 - twenty years before actual discovery of graphene. However, its basic properties, like the linearity of the spectrum, etc., were well known and widely used much earlier due to the 1947 paper by Wallace. The purpose of most of the works of the time was to describe graphite rather than graphene (see review [Castro Neto et al, 2009]).
Note that, effective fine structure constant in graphene is $\alpha_g \simeq 300\alpha = 2.19$
We will take $\alpha_g = 2$ and consider p-adic perturbation theory for graphene.

It is sixty years since Yang and Mills (1954) performed their pioneering work on gauge theories, and we have in our hands a good candidate for a theory of the strong interactions based, precisely, on a non-Abelian gauge theory, QCD.
We considered the main properties of the renormdynamics, corresponding motion equations and their solutions on the examples of QCD and other field theory models.

With the advent of any new hadron accelerator the quantities first studied are charged particle multiplicities. The multiparticle production can be described by the probability distribution P_n which is a superposition of some unknown distribution of sources F, and the Poisson distribution describing particle emission from one source. This is a typical situation in many microscopic models of multiparticle production.
Independently radiating valence quarks and corresponding negative binomial distribution presents phenomenologically preferable mechanism of hadronization in multiparticle production processes.

Let us consider $l-$particle semi-inclusive distribution

$$F_l(n,q) = \frac{d^l \sigma_n}{\bar{d}q_1 ... \bar{d}q_l} = \frac{1}{n!} \int \prod_{i=1}^{n} \bar{d}q_i' \delta(p_1 + p_2 - \Sigma_{i=1}^{l} q_i - \Sigma_{i=1}^{n} q_i')$$
$$\cdot |M_{n+l+2}(p_1, p_2, q_1, ..., q_l, q_1', ..., q_n'; g(\mu), m(\mu)), \mu)|^2,$$
$$\bar{d}p \equiv \frac{d^3 p}{E(p)}, \; E(p) = \sqrt{p^2 + m^2}. \tag{202}$$

From the renormdynamic equation

$$DM_{n+l+2} = \frac{\gamma}{2}(n + l + 2) M_{n+l+2}, \tag{203}$$

we obtain

$$\begin{aligned}
&DF_l(n,q) = \gamma(n+l+2)F_l(n,q),\\
&DF_l(q) = \gamma(<n>+l+2)F_l(q),\\
&D<n^k(q)>=\gamma(<n^{k+1}(q)>-<n^k(q)><n(q)>),\\
&DC_k = \gamma<n(q)>(C_{k+1}-C_k(1+k(C_2-1)))\\
&F_l(q) \equiv \frac{d^l\sigma}{\bar{d}q_1...\bar{d}q_l} = \sum_n \frac{d^l\sigma_n}{\bar{d}q_1...\bar{d}q_l},\ <n^k(q)>=\frac{\sum_n n^k d^l\sigma_n/\bar{d}q^l}{\sum_n d^l\sigma_n/\bar{d}q^l}\\
&C_k = \frac{<n^k(q)>}{<n(q)>^k}
\end{aligned} \tag{204}$$

From dimensional considerations, the following combination of cross sections [Koba et al, 1972] must be universal function

$$< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \quad (205)$$

Corresponding relation for the inclusive cross sections is [Matveev et al, 1976].

$$< n(p) > \frac{d\sigma_n}{d\bar{p}} / \frac{d\sigma}{d\bar{p}} = \Psi(\frac{n}{< n(p) >}). \quad (206)$$

Indeed, let us define $n-$dimension of observables [Makhaldiani, 1980]

$$[n] = 1, \ [\sigma_n] = -1, \sigma = \Sigma_n \sigma_n, \ [\sigma] = 0, \ [< n >] = 1. \quad (207)$$

The following expression does not depend on any dimensional quantities and must have a corresponding universal form

$$P_n =< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \quad (208)$$

For any discrete variable n, if the change of summation on the integration is good approximation, we can invent corresponding dimension and use dimensional counting.

Let us find an explicit form of the universal functions using renormdynamic equations. From the definition of the moments we have

$$C_k = \int_0^\infty dx x^k \Psi(x), \tag{209}$$

so they are universal parameters,

$$\begin{aligned} &DC_k = 0 \Rightarrow C_{k+1} = (1 + k(C_2 - 1))C_k \Rightarrow \\ &C_k = (1 + (k-1)(C_2 - 1))...(1 + 2(C_2 - 1))C_2. \end{aligned} \tag{210}$$

Now we can invert momentum transform and find (see [Makhaldiani, 1980]) universal functions [Ernst, Schmit, 1976], [Darbaidze et al, 1978].

$$\begin{aligned} &\Psi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn z^{-n-1} C_n = \frac{c^c}{\Gamma(c)} z^{c-1} e^{-cz}, \\ &C_2 = 1 + \frac{1}{c} \end{aligned} \tag{211}$$

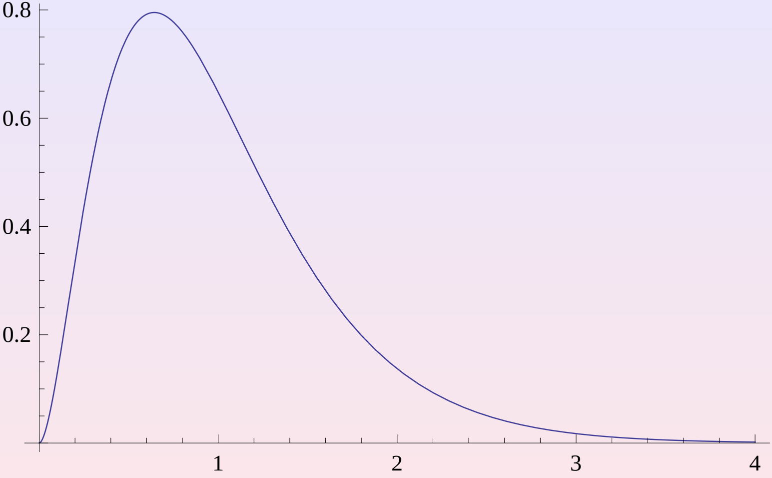

Figure: KNO distribution, $\Psi(z)$, with $c = 2.8$

The value of the parameter c can be measured from the dispersion low,

$$D = \sqrt{< n^2 > - < n >^2} = \sqrt{C_2 - 1} < n >= A < n >,$$
$$A = \frac{1}{\sqrt{c}} \simeq 0.6,\ \ c = 2.8;$$
$$(c = 3,\ \ A = 0.58) \tag{212}$$

which is in accordance with $n-$dimension counting.

We can calculate also $1/<n>$ correction to the scaling function

$$<n>\frac{\sigma_n}{\sigma}=\Psi=\Psi_0(\frac{n}{<n>})+\frac{1}{<n>}\Psi_1(\frac{n}{<n>}),$$
$$C_k=C_k^0+\frac{1}{<n>}C_k^1,$$
$$C_k^0=\int_0^\infty dx x^k\Psi_0(x),\ C_k^1=\int_0^\infty dx x^k\Psi_1(x),$$
$$\Psi_1(z)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}dn z^{-n-1}C_n^1=\frac{C_2^1c^2}{2}(z-2+\frac{c-1}{cz})\Psi_0 \quad (213)$$

The characteristic function we define as

$$\Phi(t) = \int_0^\infty dx e^{tx} \Psi(x) = (1 - t/c)^{-c}, \; Re(t) < c \tag{214}$$

For the moments of the distribution, we have

$$\Phi^{(k)}(0) = C_k = (-c)(-c-1)...(-c-k+1)(-1/c)^k = \frac{\Gamma(c+k)}{\Gamma(c)c^k} \tag{215}$$

Note that it is an infinitely divisible characteristic function, i.e.

$$\Phi(t) = (\Phi_n(t))^n, \; \Phi_n(t) = (1 - t/c)^{-c/n} \tag{216}$$

If we calculate observable(mean) value of x, we find

$$\begin{aligned} &< x >= \Phi'(0) = n\Phi(0)_n{}' = n < x >_n, \\ &< x >_n = \frac{< x >}{n} \end{aligned} \tag{217}$$

For the second moment and dispersion, we have

$$
\begin{aligned}
&< x^2 >= \Phi^{(2)}(0) = n < x^2 >_n + n(n-1) < x >_n^2, \\
&\mathrm{D}^2 =< x^2 > - < x >^2 = n(< x^2 >_n - < x >_n^2) = n\mathrm{D}_n^2 \\
&\mathrm{D}_n^2 = \frac{\mathrm{D}^2}{n} = \frac{\mathrm{D}^2}{< x >} < x >_n
\end{aligned}
\tag{218}
$$

In a sense, any Hamiltonian quantum (and classical) system can be described by infinitely divisible distributions, because in the functional integral formulation, we use the following step

$$
U(t) = e^{-itH} = (e^{-i\frac{t}{N}H})^N \tag{219}
$$

In the case of scalar field theory

$$
\begin{aligned}
L(\varphi) &= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{g}{n}\varphi^n \\
&= g^{\frac{2}{2-n}}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{1}{n}\phi^n\right)
\end{aligned} \tag{220}
$$

so, to the constituent field ϕ_N corresponds higher value of the coupling constant,

$$
g_N = gN^{\frac{n-2}{2}} \tag{221}
$$

For weak nonlinearity, $n = 2 + 2\varepsilon, d = 2/\varepsilon + 2, g_N = g(1 + \varepsilon \ln N + O(\varepsilon^2))$

Negative binomial distribution (NBD) is defined as

$$P(n) = \frac{\Gamma(n+r)}{n!\Gamma(r)} p^n (1-p)^r, \ \sum_{n\geq 0} P(n) = 1, \tag{222}$$

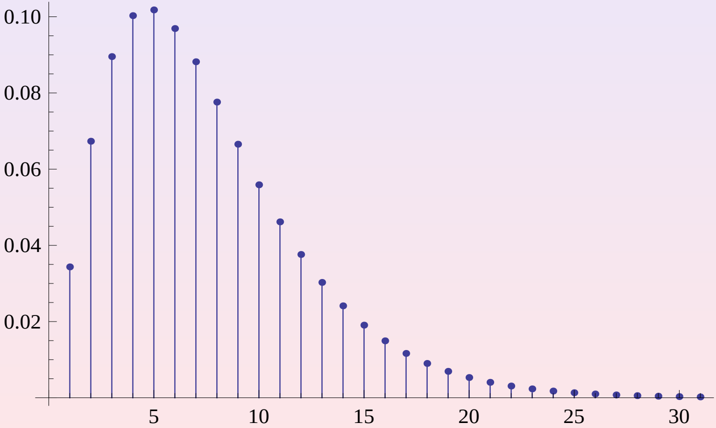

Figure: $P(n), r = 2.8, p = 0.3, < n >= 6$

NBD provides a very good parametrization for multiplicity distributions in e^+e^- annihilation; in deep inelastic lepton scattering; in proton-proton collisions; in proton-nucleus scattering.

Hadronic collisions at high energies (LHC) lead to charged multiplicity distributions whose shapes are well fitted by a single NBD in fixed intervals of central (pseudo)rapidity η [ALICE, 2010].

It is interesting to understand how NBD fits such a different reactions?

Let us consider NBD for normed topological cross sections

$$\frac{\sigma_n}{\sigma} = P(n) = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(1+\frac{k}{<n>})^{-n}(1+\frac{<n>}{k})^{-k}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{<n>}{<n>+k})^n(\frac{k}{k+<n>})^k,$$
$$= \frac{\Gamma(k+n)}{\Gamma(k)n!}\frac{(\frac{k}{<n>})^k}{(1+\frac{k}{<n>})^{k+n}},$$
$$r = k > 0,\ p = \frac{<n>}{<n>+k}. \tag{223}$$

The generating function for NBD is

$$F(h) = (1+\frac{<n>}{k}(1-h))^{-k} = (1+\frac{<n>}{k})^{-k}(1-ah))^{-k},$$
$$a = p = \frac{<n>}{<n>+k}. \tag{224}$$

Indeed,

$$
\begin{aligned}
(1-ah))^{-k} &= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1} e^{-t(1-ah)} \\
&= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1} e^{-t}\sum_0^\infty \frac{(tah)^n}{n!} \\
&= \sum_0^\infty \frac{\Gamma(n+k)a^n}{\Gamma(k)n!}h^n, \\
P(n) &= (1+\frac{<n>}{k})^{-k}\frac{\Gamma(n+k)}{\Gamma(k)n!}(\frac{<n>}{<n>+k})^n \\
&= \frac{k^k\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)}(<n>+k)^{-(n+k)} <n>^n \\
&= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)} \qquad (225)
\end{aligned}
$$

The Bose-Einstein distribution is a special case of NBD with $k=1$.

If k is negative, the NBD becomes a positive binomial distribution, narrower than Poisson (corresponding to negative correlations).
For negative (integer) values of $k = -N$, we have Binomial GF

$$F_{bd} = (1 + \frac{<n>}{N}(h-1))^N = (a+bh)^N,\ a = 1 - \frac{<n>}{N}, b = \frac{<n>}{N},$$
$$P_{bd}(n) = C_N^n(\frac{<n>}{N})^n(1 - \frac{<n>}{N})^{N-n} \qquad (226)$$

(In a sense) we have a (quantum) spectrum for the parameter k, which contains any (positive) real values and (with finite number of states) the negative integer values, $(0 \leq n \leq N)$
From the generating function we have

$$<n^2> = (\frac{hd}{dh})^2 F(h)|_{h=1} = \frac{k+1}{k} <n>^2 + <n>, \qquad (227)$$

for dispersion we obtain

$$D = \sqrt{<n^2> - <n>^2} = \frac{1}{\sqrt{k}} <n> (1 + \frac{k}{<n>})^{1/2}$$
$$= \frac{1}{\sqrt{k}} <n> + \frac{\sqrt{k}}{2} + O(1/<n>), \qquad (228)$$

So, the dispersion low for KNO and NBD distributions are the same, with $c = k$, for high values of the mean multiplicity.
The factorial moments of NBD,

$$F_m = (\frac{d}{dh})^m F(h)|_{h=1} = \frac{< n(n-1)...(n-m+1) >}{< n >^m} = \frac{\Gamma(m+k)}{\Gamma(m)k^m}, \quad (229)$$

and usual normalized moments of KNO (215) coincides.

Using fractal calculus (see e.g. [Makhaldiani, 2003]),

$$
\begin{aligned}
D_{0,x}^{-\alpha} f &= \frac{|x|^\alpha}{\Gamma(\alpha)} \int_0^1 |1-t|^{\alpha-1} f(xt)dt, = \frac{|x|^\alpha}{\Gamma(\alpha)} B(\alpha,\ \partial x) f(x) \\
&= |x|^\alpha \frac{\Gamma(\partial x)}{\Gamma(\alpha+\partial x)} f(x),\ f(xt) = t^{x\frac{d}{dx}} f(x). \qquad (230)
\end{aligned}
$$

we can define factorial and cumulant moments for any complex indexes,

$$
\begin{aligned}
&F_{-q} =< n >^q D_{0,x}^{-q} G_{NBD}(x)|_{x=0} = \frac{k^q \Gamma(k-q)}{\Gamma(k)}, \\
&K_{-q} =< n >^q D_{0,x}^{-q} \ln G_{NBD}(x)|_{x=0} = k^{q+1}\Gamma(-q), \\
&H_{-q} = \frac{\Gamma(k+1)\Gamma(-q)}{\Gamma(k-q)} \qquad (231)
\end{aligned}
$$

Let us show that NBD is a discrete distribution corresponding to the KNO scaling,

$$\lim_{<n>\to\infty} <n> P_n|_{\frac{n}{<n>}=z} = \Psi(z) \tag{232}$$

Indeed, using the following asymptotic formula

$$\Gamma(x+1) = x^x e^{-x}\sqrt{2\pi x}(1+\frac{1}{12x}+O(x^{-2})), \tag{233}$$

we find

$$<n> P_n =<n> \frac{(n+k-1)^{n+k-1}e^{-(n+k-1)}}{\Gamma(k)n^n e^{-n}}\frac{k^k}{n^k} <n> z^k e^{-k\frac{n+k}{<n>}}$$
$$= \frac{k^k}{\Gamma(k)}z^{k-1}e^{-kz} + O(1/<n>) \tag{234}$$

We can calculate also $1/<n>$ correction term to the KNO from the NBD. The answer is

$$\Psi = \frac{k^k}{\Gamma(k)}z^{k-1}e^{-kz}(1+\frac{k^2}{2}(z-2+\frac{k-1}{kz})\frac{1}{<n>}) \tag{235}$$

This form coincides with the corrected KNO (213) for $c = k$ and $C_2^1 = 1$. We have seen that KNO characteristic function (214) and NBD GF (224) have almost same form. This relation become in coincidence if

$$c = k, \ t = (h-1)\frac{<n>}{k} \tag{236}$$

Now the definition of the characteristic function (214) can be read as

$$\int_0^\infty e^{-<n>z(1-h)}\Psi(z)dz = (1 + \frac{<n>}{k}(1-h))^{-k} \tag{237}$$

which means that Poisson GF weighted by KNO distribution gives NBD GF. Because of this, the NBD is the gamma-Poisson (mixture) distribution. This is the exact and universal picture of hadronization in multiparticle production processes.

Fore high values of $x_2 = k$ the NBD distribution reduces to the Poisson distribution

$$F(x_1, x_2, h) = (1 + \frac{x_1}{x_2}(1-h))^{-x_2} \Rightarrow e^{-x_1(1-h)} = e^{-<n>} e^{h<n>}$$
$$= \sum P(n) h^n,$$
$$P(n) = e^{-<n>} \frac{<n>^n}{n!} \tag{238}$$

For the Poisson distribution

$$\frac{d^2 F(h)}{dh^2}|_{h=1} =< n(n-1) >=< n >^2,$$
$$D^2 =< n^2 > - < n >^2 =< n > . \tag{239}$$

In the case of NBD, we had the following dispersion low

$$D^2 = \frac{1}{k} < n >^2 + < n >, \tag{240}$$

which coincides with the previous expression for high values of k.

Poisson GF belongs to the class of the infinitely divisible distributions,

$$F(h, < n >) = (F(h, < n > /k))^k \tag{241}$$

For high values of $< n >$, the Poisson distribution reduces to the Gauss distribution

$$P(n) = e^{-<n>} \frac{< n >^n}{n!} \Rightarrow \frac{1}{\sqrt{2\pi < n >}} \exp(-\frac{(n- < n >)^2}{2 < n >}) \tag{242}$$

For high values of k in the integral relation (237), in the KNO function dominates the value $z_c = 1$ and both sides of the relation reduce to the Poisson GF.

A Bose-Einstein, or geometrical, distribution is a thermal distribution for single state systems. An useful property of the negative binomial distribution with parameters

$$< n >,\ k$$

is that it is (also) the distribution of a sum of k independent random variables drawn from a Bose-Einstein distribution with mean $< n > /k$,

$$\begin{aligned} P_n &= \frac{1}{< n > +1}\left(\frac{< n >}{< n > +1}\right)^n \\ &= (e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2})e^{-\beta\hbar\omega(n+1/2)},\ T = \frac{\hbar\omega}{\ln\frac{<n>+1}{<n>}} \\ &\sum_{n\geq 0} P_n = 1,\ \sum nP_n =< n >= \frac{1}{e^{\beta\hbar\omega-1}},\ T \simeq \hbar\omega < n >,\ < n >\gg 1, \\ P(x) &= \sum_n x^n P_n = (1+ < n > (1-x))^{-1}. \end{aligned} \tag{243}$$

This is easily seen from the generating function in (224), remembering that the generating function of a sum of independent random variables is the product of their generating functions.

Indeed, for

$$n = n_1 + n_2 + ... + n_k, \tag{244}$$

with n_i independent of each other, the probability distribution of n is

$$P_n = \sum_{n_1,...,n_k} \delta(n - \sum n_i) p_{n_1}...p_{n_k},$$
$$P(x) = \sum_n x^n P_n = p(x)^k \tag{245}$$

This has a consequence that an incoherent superposition of N emitters that have a negative binomial distribution with parameters $k, < n >$ produces a negative binomial distribution with parameters $Nk, N < n >$.
So, for the GF of NBD we have (N=2)

$$F(k, < n >)F(k, < n >) = F(2k, 2 < n >) \tag{246}$$

And more general formula (N=m) is

$$F(k, < n >)^m = F(mk, m < n >) \tag{247}$$

We can put this equation in the closed nonlocal form

$$Q_q F = F^q, \tag{248}$$

where

$$Q_q = q^D, \quad D = \frac{kd}{dk} + \frac{<n>d}{d<n>} = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2} \tag{249}$$

Note that temperature defined in (243) gives an estimation of the Glukvar temperature when it radiates hadrons. If we take $\hbar\omega = 100MeV$, to $T \simeq T_c \simeq 200MeV$ corresponds $<n> \simeq 1.5$ If we take $\hbar\omega = 10MeV$, to $T \simeq T_c \simeq 200MeV$ corresponds $<n> \simeq 20$. A singular behavior of $<n>$ may indicate corresponding phase transition and temperature. At that point we estimate characteristic quantum $\hbar\omega$.
We see that universality of NBD in hadron-production is similar to the universality of black body radiation.

Let us imagine space-time development of the the multiparticle process and try to describe it by some (phenomenological) dynamical equation. We start to find the equation for the Poisson distribution and than naturally extend them for the NBD case.
Let us define an integer valued variable $n(t)$ as a number of events (produced particles) at the time $t,\ n(0) = 0$. The probability of event $n(t),\ P(t,n)$, is defined from the following motion equation

$$P_t \equiv \frac{\partial P(t,n)}{\partial t} = r(P(t,n-1) - P(t,n)),\ n \geq 1$$
$$P_t(t,0)) = -rP(t,0),$$
$$P(t,n) = 0,\ n < 0, \tag{250}$$

so

$$P(t,0) \equiv P_0(t) = e^{-rt},$$
$$P(t,n) = Q(t,n)P_0(t),$$
$$Q_t(t,n) = rQ(t,n-1),\ Q(t,0) = 1. \tag{251}$$

To solve the equation for Q, we invent its generating function

$$F(t,h) = \sum_{n>0} h^n Q(t,n), \tag{252}$$

and solve corresponding equation

$$F_t = rhF,\ F(t,h) = e^{rth} = \sum h^n \frac{(rt)^n}{n!},\ Q(t,n) = \frac{(rt)^n}{n!}, \tag{253}$$

so

$$P(t,n) = e^{-rt}\frac{(rt)^n}{n!} \tag{254}$$

is the Poisson distribution.
If we compare this distribution with (242), we identify $< n >= rt$, as if we have a free particle motion with velocity r and the distance is the mean multiplicity. This way we have a connection between n-dimension of the multiplicity and the usual dimension of trajectory.
As the equation gives right solution, its generalization may give more general distribution, so we will generalize the equation (250). For this, we put the equation in the closed form

$$\begin{aligned} P_t(t,n) &= r(e^{-\partial_n} - 1)P(t,n) \\ &= \sum_{k\geq 1} D_k \partial^k P(t,n),\ D_k = (-1)^k \frac{r}{k!}, \end{aligned} \tag{255}$$

where the $D_k,\ k \geq 1$, are generalized diffusion coefficients.

For other values of the coefficients, we will have other distributions. For mean square deviation of the trajectory we have

$$< (x-\bar{x})^2 >=< x^2 > - < x >^2 \equiv D(x)^2 \sim t^{2/d_f}, \tag{256}$$

where d_f is fractal dimension. For smooth classical trajectory of particles we have $d_f = 1$; for free stochastic, Brownian, trajectory, all diffusion coefficients are zero but D_2, we have $d_f = 2$. In the case of Poisson process we have,

$$D(n)^2 =< n^2 > - < n >^2 \sim t,\ d_f = 2. \tag{257}$$

In the case of the NBD and KNO distributions

$$D(n)^2 \sim t^2,\ d_f = 1. \tag{258}$$

As we have seen, rasing k, KNO reduce to the Poisson, so we have a dimensional (phase) transition from the phase with dimension 1 to the phase with dimension 2. It is interesting, if somehow this phase transition is connected to the other phase transitions in strong interaction processes.

For the Poisson distribution GF is solution of the following equation,

$$\dot{F} = -r(1-h)F, \tag{259}$$

For the NBD corresponding equation is

$$\dot{F} = \frac{-r(1-h)}{1+\frac{rt}{k}(1-h)}F = -R(t)F, \;\; R(t) = \frac{r(1-h)}{1+\frac{rt}{k}(1-h)}. \tag{260}$$

If we change the time variable as $t = T^{d_f}$, we reduce the dispersion low from general fractal to the NBD like case. Corresponding transformation for the evolution equation is

$$F_T = -d_f T^{d_f-1} R(T^{d_F})F, \tag{261}$$

we ask that this equation coincides with NBD motion equation, and define rate function $R(T)$

$$d_f T^{d_f-1} R(T^{d_F}) = \frac{r(1-h)}{1+\frac{rT}{k}(1-h)} \tag{262}$$

The following equation defines a production processes with fractal dimension d_F

$$F_t = -R(t)F,\ R(t) = \frac{r(1-h)}{d_F t^{\frac{d_F-1}{d_F}}\left(1+\frac{rt^{1/d_F}}{k}(1-h)\right)} \tag{263}$$

Motion equations of physics (applied mathematics in general) connect different observable quantities and reduce the number of independently measurable quantities. More fundamental equation contains less number of independent quantities. When (before) we solve the equations, we invent dimensionless invariant variables, than one solution can describe all of the class of phenomena.
In the z - Scaling (zS) approach to the inclusive multiparticle distributions (MPD) (see, e.g. [Tokarev, Zborovsky, 2007]), different inclusive distributions depending on the variables $x_1, ...x_n$, are described by universal function $\Psi(z)$ of fractal variable z,

$$z = x_1^{-\alpha_1} ... x_n^{-\alpha_n}. \tag{264}$$

It is interesting to find a dynamical system which generates this distributions and describes corresponding MPD.

We can find a good function if we know its derivative. Let us consider the following RD like equation

$$z\frac{d}{dz}\Psi = V(\Psi),$$
$$\int_{\Psi(z_0)}^{\Psi(z)} \frac{dx}{V(x)} = \ln\frac{z}{z_0} \tag{265}$$

As a dimensionless physical quantity $\Psi(z)$ may depend only on the running coupling constant $g(\tau),\ \tau = \ln z/z_0$

$$z\frac{d}{dz}\Psi = \dot{\Psi} = \frac{d\Psi}{dg}\beta(g) = U(g) = U(f^{-1}(\Psi)) = V(\Psi),$$
$$\Psi(\tau) = f(g(\tau)),\ g = f^{-1}(\Psi(\tau)) \qquad (266)$$

According to the paper [Tokarev, Zborovsky, 2007], for high values of z, $\Psi(z) \sim z^{-\beta}$; for small z, $\Psi(z) \sim$ const.
So, for high z,

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = -\beta\Psi(z); \tag{267}$$

for smaller values of z, $\Psi(z)$ rise and we expect nonlinear terms in $V(\Psi)$,

$$V(\Psi) = -\beta\Psi + \gamma\Psi^2. \tag{268}$$

With this function, we can solve the equation for Ψ and find

$$\Psi(z) = \frac{1}{\frac{\gamma}{\beta} + cz^{\beta}}. \tag{269}$$

Let us consider more general potential V

$$z\frac{d}{dz}\Psi = V(\Psi) = -\beta\Psi(z) + \gamma\Psi(z)^{1+n} \tag{270}$$

Corresponding solution for Ψ is

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz^{n\beta})^{\frac{1}{n}}} \tag{271}$$

More general solution contains three parameters and may better describe the data of inclusive distributions.

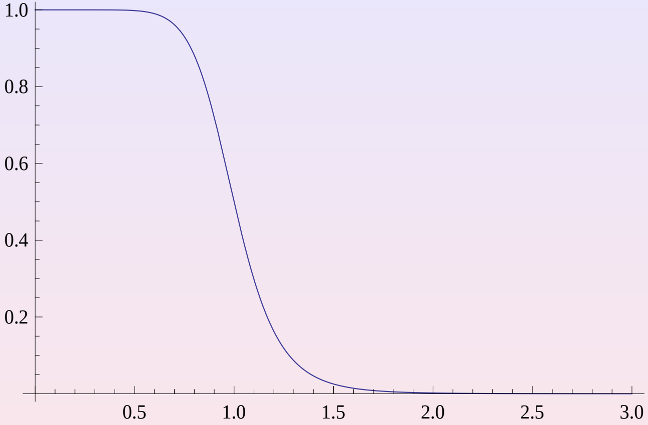

Figure: z-scaling distribution $\Psi(z, 9, 9, 1, 1)$

In the case of $n = 1$ we reproduce the previous solution.
Another "natural" case is $n = 1/\beta$,

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz)^{\beta}} \quad (272)$$

In this case, we can absorb (interpret) the combined parameter by shift and scaling

$$z \to \frac{1}{c}(z - \frac{\gamma}{\beta}) \quad (273)$$

Another interesting point of view is to predict the value of β

$$\beta = \frac{1}{n} = 0.5; 0.33; 0.25; 0.2; ..., \ \ n = 2, 3, 4, 5, ... \quad (274)$$

For experimentally suggested value $\beta \simeq 9, n = 0.11$
The three parameter function is restricted by the normalization condition

$$\int_0^{\infty} \Psi(z)dz = 1,$$
$$B(\frac{\beta - 1}{\beta n}, \frac{1}{\beta n}) = (\frac{\beta}{\gamma})^{\frac{\beta-1}{\beta n}} \frac{\beta n}{c^{\beta n}}, \quad (275)$$

When $\beta n = 1$, we have

$$c = (\beta - 1)(\frac{\beta}{\gamma})^{\beta - 1} \tag{276}$$

If $\beta n = 1$ and $\beta = \gamma$, than $c = \beta - 1$.
In general

$$c^{\beta n} = (\frac{\beta}{\gamma})^{\frac{\beta - 1}{\beta n}} \frac{\beta n}{B(\frac{\beta - 1}{\beta n}, \frac{1}{\beta n})} \tag{277}$$

The dimension of the space(-time) is the model dependent concept. E.g. for the fundamental bosonic string model (in flat space-time) the dimension is 26; for superstring model the dimension is 10 [Kaku, 2000].
Let us imagine that we have some action-functional formulation with the fundamental motion equation

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = V(\Psi) = -\beta\Psi + \gamma\Psi^{1+n}. \tag{278}$$

Than, the corresponding Lagrangian contains the following mass and interaction parts

$$-\frac{\beta}{2}\Psi^2 + \frac{\gamma}{2+n}\Psi^{2+n} \tag{279}$$

The action gives renormalizable (effective quantum field theory) model when

$$d + 2 = \frac{2N}{N-2} = \frac{2(2+n)}{n} = 2 + \frac{4}{n} = 2 + 4\beta, \tag{280}$$

so, measuring the parameter β inside hadronic and nuclear matters, we find corresponding (fractal) dimension.

From fundamental equation we obtain

$$(z\frac{d}{dz})^2\Psi \equiv \ddot{\Psi} = V'(\Psi)V(\Psi) = \frac{1}{2}(V^2)' \\ = \beta^2\Psi - \beta\gamma(n+2)\Psi^{n+1} + \gamma^2(n+1)\Psi^{2n+1} \quad (281)$$

Corresponding action Lagrangian is

$$L = \frac{1}{2}\dot{\Psi}^2 + U(\Psi),\ U = \frac{1}{2}V^2 = \frac{1}{2}\Psi^2(\beta - \gamma\Psi^n)^2 \\ = \frac{\beta^2}{2}\Psi^2 - \beta\gamma\Psi^{2+n} + \frac{\gamma^2}{2}\Psi^{2+2n} \quad (282)$$

This potential, $-U$, has two maximums, when $V = 0$, and minimum, when $V' = 0$, at $\Psi = 0$ and $\Psi = (\beta/\gamma)^{1/n}$, and $\Psi = (\beta/(n+1)\gamma)^{1/n}$, correspondingly.

We define time-space-scale field $\Psi(t, x, \eta)$, where $\eta = \ln z-$ is scale coordinate variable, with corresponding action functional

$$A = \int dt d^d x d\eta(\frac{1}{2}g^{ab}\partial_a\Psi\partial_b\Psi + U(\Psi)) \quad (283)$$

The renormalization constraint for this action is

$$N = 2 + 2n = \frac{2(2+d)}{2+d-2} = 2 + \frac{4}{d},\ dn = 2,\ d = 2/n = 2\beta. \tag{284}$$

So we have two models for spase-time dimension, (280) and (284),

$$d_1 = 4\beta;\ d_2 = 2\beta \tag{285}$$

The coordinate η characterise (multiparticle production) physical process at the (external) space-time point (x,t). The dimension of the space-time inside hadrons and nuclei, where multiparticle production takes place is

$$d + 1 = 1 + 2\beta \tag{286}$$

Note that this formula reminds the dimension of the spin s state, $d_s = 2s + 1$. If we take $\beta(= s) = 0; 1/2; 1; 3/2; 2; ...$ We will have $d + 1 = 1; 2; 3; 4; 5; ...$

O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large N field theories, string theory and gravity, *Phys. Rept.* **323** (2000) 183, hep-th/9905111.

K. Aamodt et al. [ALICE collaboration] *Eur. Phys. J.* **C65** (2010) 111 [arXiv:1004.3034], [arXiv:1004.3514].

M. Anselmino, B.L. Ioffe and E. Leader, Sov. J. Nucl. 49, 136, (1989)

H. Aref, Ann. Rev. Fluid Mech. **15** (1983) 345.

V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York 1978.

J. Bagger, N. Lambert, *Modeling multiple M2's*, Phys. Rev. D **75** (2007) 045020; [arXiv:hep-th/0611108].

V.V. Balandin, N.I. Golubeva, *Hamiltonian Methods for the Study of Polarized Proton Beam Dynamics in Accelerators and Storage Rings*, [arXiv://arxiv.org/abs/physics/9903032v1].

V. Bargmann, L. Michel, V.L. Telegdi, *Precession of the polarization of particles moving in a homogeneous electromagnetic field*, Phys. Rev. Lett., **2(10)** (1959) 435.

G. Benenti, G. Casati, G. Strini, *Principles of quantum computation and information*, Vol. I: Basic concepts, World Scientific, Singapore 2004; Vol. II: *Basic tools and special topics* World Scientific, Singapore 2007.

J. Beringer et al. (Particle Data Group) *Review of Particle Physics*, Phys. Rev. D **86** (2012) 010001 [1528 pages]

J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

F.A. Berezin, *Introduction to Superanalysis*, Reidel, Dordrecht 1987.

F.A. Berezin, M.S. Marinov, Ann. Phys. (N.Y.) **104** (1977) 336.

N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, New York 1959.

S. Brodsky, G. Farrar, Phys. Rev. Lett. **31** 1153 (1973).

Stanley J. Brodsky, Guy F. de Tèramond, Alexandre Deur, Nonperturbative QCD Coupling and its β function from Light-Front Holography, PRD 81,096010 (2010), arXiv:1002.3948

V. D. Burkert, Phys. Rev. D 63, 097904 (2001).

C. Buttin, C.R. Acad. Sci. Paris. **269** (1969) 87.

A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov and A. K. Geim, The electronic properties of graphene, Rev. Mod. Phys. 81 , 109-162 (2009).

W. E. Caswell, Phys. Rev. Lett. 33 (1974) 244.

J.C. Collins, *Renormalization*, Cambrige Univ. Press, London 1984.

Ya.Z. Darbaidze, N.V. Makhaldiani, A.N. Sisakian, L.A. Slepchenko, *TMF* **34** (1978) 303.

D. Diakonov, Prog. Par. Nucl. Phys. 51 (2003) 173.

P.A.M. Dirac, *Proc. Roy, Soc.* **A167** (1938) 148.

A.D. Dolgov, Light millicharged particles and large scale cosmic magnetic fields, arXiv:1404.5820v1 [astro-ph.CO] 23 Apr 2014

S. D. Drell, A. C. Hearn, *Exact Sum Rule For Nucleon Magnetic Moments, Phys. Rev. Lett.* **16** (1966) 908.

E. Egorian and O. V. Tarasov, Theor. Math. Phys. 41 (1979) 863 [Teor. Mat. Fiz. 41 (1979) 26]

W. Ernst, I. Schmitt, *Nuovo Cim.* **A33** (1976) 195.

L.D. Faddeev, R. Jackiw, Phys.Rev.Lett. **60** (1988) 1692.

L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer, Berlin 1990.

I. V. Fialkovsky, D. V. Vassilevich , Quantum Field Theory in Graphene, arXiv:1111.3017

C. S. Fischer and H. Gies, JHEP 0410 (2004) 048.

V. Flambaum, I. Khriplovich and O. Sushkov, Nucl.Phys. A449, p. 750 (1986).

S.B. Gerasimov, *A Sum Rule for Magnetic Moments and Damping of the Nucleon Magnetic Moment in Nuclei*, *J.Nucl.Phys.(USSR)* **2** (1965) 598.

V. L. Ginzburg, *Theoretical Physics and Astrophysics*, Pergamon, New York 1979.

M.L. Goldberger, S.B. Treiman, Phys. Rev. **110** 1178 (1958).

D.J. Gross, F. Wilczek, Phys. Rev. Lett. **30** 1343 (1973).

S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109];

A. Gustavsson, *Algebraic structures on parallel M2-branes*, Nucl. Phys. B.**811** (2009) 66; [arXiv:0709.1260 [hep-th]].

G. Gutin, A.P. Punnen (Eds), *The Traveling Salesman Problem and Its Variations*, Combinatorial Optimization Series, Kluwer, Boston 2002.

W. Heisenberg, *Introduction to the Unified field Theory of Elementary particles,* Interscience Publishers, London 1966.

G. 't Hooft, report at the *Marseille Conference on Yang-Mills Fields*, 1972.

G. 't Hooft, *Nucl.Phys.* **B 61** (1973) 455.

Ernst-Michael Ilgenfritz, Axel Maas, Topological aspects of G2 Yang-Mills theory, arXiv:1210.5963.

J.D. Jackson, *Classical Electrodynamics, 3rd ed.* JohnWiley & Sons, Inc. New York 1999.

X. Ji, J.Osborne, J.Phys. G27 127 (2001).

D. R. T. Jones, Nucl. Phys. B 75 (1974) 531;

M. Kaku, *Strings, Conformal Fields, and M-Theory*, Springer, New York 2000.

A. L. Kataev, S. A. Larin, Analytical five-loop expressions for the renormalization group QED β-function in different renormalization schemes, Pisma v ZhETF, vol.96, iss. 1, (2012), pp.64-67

D.I. Kazakov, *Supersymmetric Generalization of the Standard Model of Fundamental Interactions*, Textbook, JINR Dubna 2004.

D.I. Kazakov, D.V. Shirkov, Fortschr. d. Phys. **28** 447 (1980).

D.I. Kazakov, L.R. Lomidze, N.V. Makhaldiani, A.A. Vladimirov, *Ultraviolet Asymptotics in Renormalizable Scalar Theories*, JINR Communications, **E2-8085**, Dubna 1974.

A.Yu. Kitaev, A. Shen, M.N. Vyalyi, *Classical and Quantum Computation*, American Mathematical Society, 2002.

Z. Koba, H.B. Nielsen, P. Olesen, *Nucl. Phys.* **B40** (1972) 317.

N.V. Makhaldiani, *Approximate methods of the field theory and their applications in physics of high energy, condensed matter, plasma and hydrodynamics*, Dubna 1980.

S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303 (1993) 334.

C. Lerche and L. von Smekal, Phys. Rev. D 65 (2002) 125006.

N.M. Makhaldiani, *Computational Quantum Field Theory*, JINR Communication, **P2-86-849**, Dubna 1986.

N.V. Makhaldiani, *A New Approach to the Problem of Space Compactification,*
JINR Communications, **P2-87-306**, Dubna 1987.

N.V. Makhaldiani, *Number Fields Dynamics and the Compactification of Space Problem in the Unified Theories of Fields and Strings*,
JINR Communications, **P2-88-916**, Dubna 1988.

N. Makhaldiani, *The System of Three Vortexes of Two-Dimensional Ideal Hydrodynamics as a New Example of the (Integrable) Nambu-Poisson Mechanics*, JINR Communications **E2-97-407**, Dubna 1997; [arXiv:solv-int/9804002].

N. Makhaldiani, *The Algebras of the Integrals of Motion and Modified Bochner-Killing-Yano Structures of the Point particle Dynamics*,
JINR Communications **E2-99-337**, Dubna 1999.

N. Makhaldiani, *New Hamiltonization of the Schrödinger Equation by Corresponding Nonlinear Equation for the Potential*,
JINR Communications, **E2-2000-179**, Dubna 2000.

N. Makhaldiani, *How to Solve the Classical Problems on Quantum Computers*,
JINR Communications, **E2-2001-137**, Dubna 2001.

N. Makhaldiani, *Classical and Quantum Problems for Quanputers*, [arXiv:quant-ph/0210184].

N. Makhaldiani, *Adelic Universe and Cosmological Constant*,
JINR Communications, **E2-2003-215** Dubna 2003, [arXiv:hep-th/0312291].

N. Makhaldiani, *Nambu-Poisson dynamics of superintegrable systems*, *Atomic Nuclei*, **70** (2007) 564.

N. Makhaldiani, *Theory of Quanputers*, Sovremennaia Matematica i ee Prilozhenia, **44** (2007) 113; Journal of Mathematical Sciences, **153** (2008) 159.

N.V. Makhaldiani, *Renormdynamics and Scaling Functions*, in Proc. of the *XIX International Baldin Seminar on High Energy Physics Problems* eds. A.N.Sissakian, V.V.Burov, A.I.Malakhov, S.G.Bondartenko, E.B.Plekhanov, Vol.II, p. 175, Dubna 2008.

N.V. Makhaldiani, *Renormdynamics, multiparticle production, negative binomial distribution and Riemann zeta function*, [arXiv:1012.5939v1 [math-ph]] 24 Dec 2010.

N. Makhaldiani, *Fractal Calculus (H) and some Applications*, *Physics of Particles and Nuclei Letters*, **8** 325 (2011).

N. Makhaldiani, *Regular method of construction of the reversible dynamical systems and their linear extensions - Quanputers*, Atomic Nuclei, **74** (2011) 1040.

Nugzar Makhaldiani, *Nambu-Poisson Dynamics with Some Applications*,
Physics of Particles and Nuclei, **43** (2012) 703.

Nugzar Makhaldiani, *Renormdynamics, coupling constant unification and universal properties of the multiparticle production*, XXI International Baldin Seminar on High Energy Physics Problems, September 10-15, 2012, PoS(Baldin ISHEPP XXI)068.

N. V. Makhaldiani, *Renormdynamics, Multiparticle Production, Negative Binomial Distribution, and Riemann Zeta Function*, Physics of Atomic Nuclei, **76** 1169 (2013).

N. Makhaldiani, O. Voskresenskaya, *On the correspondence between the dynamics with odd and even brackets and generalized Numbu's mechanics,*
JINR Communications, **E2-97-418**, Dubna 1997.

J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

V. Matveev, R. Muradyan, A. Tavkhelidze, Lett. Nuovo Cimento **7** 719 (1973).

V.A. Matveev, A.N. Sisakian, L.A. Slepchenko, *Nucl. Phys.* **23** (1976) 432.

A.V. Meleshko, N.N. Konstantinov, *Dynamics of vortex systems*,
Naukova Dumka, Kiev 1993.

Y. Nambu, Phys.Rev. D **7** (1973) 2405.

M.A. Nielsen, I.L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge 2000.

J. M. Pawlowski, D. F. Litim, S. Nedelko and L. von Smekal, Phys. Rev. Lett. 93 (2004) 152002.

H.D. Politzer, Phys. Rev. Lett. **30** 1346 (1973).

M.C.M. Rentmeester, R.G.E. Timmermans, J.L. Friar, J.J. de Swart, Phys. Rev. Lett. **82** 4992 (1999).

T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B 400 (1997)379.

B. L. Roberts and W. J. Marciano (eds.), Lepton Dipole Moments, Advanced Series on Directions in High Energy Physics, Vol. 20 (World Scientific, 2010).

A. Sakharov, Pisma Zh.Eksp.Teor.Fiz. 5, 32 (1967), Reprinted in Sov. Phys. Usp. 34 (1991) 392-393 [Usp. Fiz. Nauk 161 (1991) No. 5 61-64].

A. Samarskii, A. Gulin, *Numerical Methods*, Nauka, Moscow 1989.

J. Schwinger, *Magnetic Charge and Quantum Field Theory, Phys. Rev.* **144** (1966) 1087.

Ya.G. Sinai, *Topics in Ergodic Theory*, Princeton University Press, Princeton NJ 1993.

O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. B 93 (1980) 429.

Norman Margolus, Tommaso Toffoli, *Cellular Automaton Machines*,
MIT Press, Cambridge 1987.

M.V. Tokarev, I. Zborovský, *Z-Scaling in the Proton-Proton Collisions at RHIC*, in *Investigations of Properties of Nuclear Matter at High Temperature and Densities, Editid by A.N. Sisakian, F.A. Soifer*, Dubna 2007

L.H. Thomas, Philos. Mag. **3** (1927) 1.

M.B. Voloshin, K.A. Ter-Martyrosian, *Gauge Theory of Elementary Particles*, Atomizdat, Moscow 1984.

S. Weinberg, AIP Conf.Proc. 272, 346 (1993).

E.T. Whittaker, *A Treatise on the Analytical Dynamics*, Cambridge 1927.

E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

Zelikin, Mikhail I. (2000), Homogeneous Spaces and the Riccati Equation in the Calculus of Variations, Berlin: Springer-Verlag

D. Zwanziger, Phys. Rev. D 65 (2002) 094039.