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THE MASS-TO-CHARGE RATIO OF SELF-GRAVITATING SCALAR FIELD CONFIGURATIONS

Main purpose:

The study of self-gravitating, static, spherically symmetric configurations of scalar and electromagnetic fields.

Research objectives:

1.) Study of all possible types of space-time causal structure in the models under consideration.

2.) Investigation of the influence of the scalar field on the mass-to-charge ratio of a black hole.

3.) Expansion of the General Relativity applicability domain.

The relevance of research:

The study of self-gravitating configurations of physical fields is an important part of research in modern theory of gravity. Mathematical models constructed on the basis of static configurations of scalar field are now actively used both for the description of elementary particles and for the study of objects on a galactic scale.

We investigate the question of possible values of the mass-to-charge ratio for a system of scalar and electromagnetic fields. Such a study is of interest from both a mathematical and a physical point of view, since it sheds light on the influence of a scalar field on the space-time geometry of a black hole.

Action, energy–momentum tensor and electromagnetic field tensor for scalar and electromagnetic fields, minimally coupled to Einstein gravity

The action (in units with G $=$ c $= h = k_e = k_B =$1):

$$\Sigma = \frac{1}{8\pi}\int\left(-\frac{1}{2}S + \mathcal{L}_\phi - \frac{1}{2}F_{ij}F^{ij}\right)\sqrt{|g|}\,d^4x\,, \quad \mathcal{L}_\phi = \varepsilon\,\langle d\phi, d\phi\rangle - 2V(\phi)$$

The metric for a static, spherically symmetric space-time:

$$g = A^2 dt\otimes dt - B^2 dr\otimes dr - C^2(d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi)\,, \quad (1)$$

$A\,, B\,, C$ – **functions of coordinates** $r\,, t\,.$

The energy–momentum tensor

$$T \, = \, T_{(\phi)} \, + \, T_{(em)} \tag{2}$$

for scalar field:

$$T_{(\phi)ij} \, = \, \frac{1}{8\pi} \left(2 \, \frac{\delta \, \mathcal{L}_\phi}{\delta \, g^{ij}} \, - \, \mathcal{L}_\phi \, g_{ij} \right) \tag{3}$$

for electromagnetic field:

$$T_{(em)ij} \, = \, -\frac{1}{4\pi} \, g_{ik} \, F_{jl} F^{kl} \, + \, \frac{1}{16\pi} \, g_{ij} \, F_{kl} F^{kl} \tag{4}$$

The system of Einstein equations:

$$\mathcal{R} - \frac{1}{2} S\, g \,=\, 8\,\pi\, T \tag{5}$$

The Klein–Gordon equation:

$$\Box\,\phi \,+\, \varepsilon\, V'_\phi \,=\, 0 \tag{6}$$

or, in coordinate form:

$$\frac{1}{\sqrt{|g|}}\,\partial_i\,(\sqrt{|g|}\,g^{ij}\,\partial_j\phi) \,+\, \varepsilon\, V'_\phi \,=\, 0 \tag{7}$$

$|g| \,=\, |det\,(g_{ij})|$

Maxwell equation:

$$d\, *\, F \,=\, 0\,, \tag{8}$$

where $*$ is the Hodge star operator defined on 2-forms by the following formula

$$\eta \mapsto\; *\eta \,=\, \frac{1}{2}\mathcal{E}\cdot\eta\,,\;\; \mathcal{E} = e^0\wedge e^1\wedge e^2\wedge e^3,$$

the dot indicates a complete metric contraction.

An orthonormal basis of vector fields:

$$e_0 = \frac{1}{A}\partial_t\,, \quad e_1 = \frac{1}{B}\partial_r\,, \quad e_2 = \frac{1}{C}\partial_\theta\,, \quad e_3 = \frac{1}{C\sin\theta}\partial_\varphi\,.$$

$$C_{(0)} \equiv \mathrm{e}_0 C = (1/A)\partial_t C\,,\ \phi_{(1)} \equiv \mathrm{e}_1\phi = (1/B)\partial_r\phi$$

– the directional derivative along the basis vector fields.

2-form of electromagnetic field:

$$F_{01}\,C^2 \;=\; const \;=\; q \;\;\Longrightarrow\; F^{01} \;=\; -\,F^{10} \;=\; F_{10} \;=\; -\,F_{01}\,, \qquad (9)$$

q – **the charge**.

References

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System of equations:

$$-2\frac{C_{(1)(1)}}{C}+2\frac{B_{(0)}C_{(0)}}{BC}-\frac{C_{(1)}^2-C_{(0)}^2-1}{C^2} = \varepsilon(\phi_{(1)}^2+\phi_{(0)}^2)+2\,V + \frac{q^2}{C^4}, \tag{10}$$

$$-2\frac{C_{(0)(0)}}{C}+2\frac{A_{(1)}C_{(1)}}{AC}+\frac{C_{(1)}^2-C_{(0)}^2-1}{C^2} = \varepsilon(\phi_{(1)}^2+\phi_{(0)}^2)-2\,V - \frac{q^2}{C^4}, \tag{11}$$

$$\frac{A_{(1)(1)}}{A}-\frac{B_{(0)(0)}}{B}+\frac{C_{(1)(1)}}{C}-\frac{C_{(0)(0)}}{C}+\frac{A_{(1)}C_{(1)}}{AC}-\frac{B_{(0)}C_{(0)}}{BC} = \\ = \varepsilon(\phi_{(0)}^2-\phi_{(1)}^2)-2\,V + \frac{q^2}{C^4}, \tag{12}$$

$$-2\frac{C_{(0)(1)}}{C}+2\frac{B_{(0)}C_{(1)}}{BC} \equiv -2\frac{C_{(1)(0)}}{C}+2\frac{A_{(1)}C_{(0)}}{AC} = 2\,\varepsilon\phi_{(0)}\phi_{(1)}. \tag{13}$$

The Klein–Gordon equation:

$$\phi_{(0)(0)}-\phi_{(1)(1)}+\phi_{(0)}\frac{(BC^2)_{(0)}}{BC^2}-\phi_{(1)}\frac{(AC^2)_{(1)}}{AC^2}+\varepsilon\,V'_{\phi} = 0. \tag{14}$$

A general asymptotically flat solution of the inverse problem:

$$f(C) = 2\,C^2 \mathrm{e}^{-2F} \int_C^\infty \frac{(Q - 3\,m - 2\,q^2\,P)\,\mathrm{e}^F}{C^4} dC \tag{15}$$

$$V(C) = \frac{1}{2\,C^{\,2}} \left(1 - f - \varepsilon\,C^{\,2}\,\phi^{\vee\,2}\,f - C\,f^{\vee} - \frac{q^2}{C^2}\right), \tag{16}$$

where

$$F(C) = -\int_C^\infty \varepsilon\,\phi^{\vee\,2}\,C\,dC\,, \quad P(C) = -\int_C^\infty \frac{\mathrm{e}^F}{C^2} dC \tag{17}$$

$$Q(C) = C + \int_C^\infty \left(1 - \mathrm{e}^F\right) dC = C + o(1),\;\; C \to \infty\,. \tag{18}$$

The parameter m is the Schwarzschild mass. Metric:

$$ds^2 = e^F f\,dt^2 - \frac{dC^2}{f} - C^2(d\theta^2 + \sin^2\theta\,d\varphi^2)\,.$$

A three-parameter family of exact asymptotically flat solutions:

$$\phi = a^2\,\tilde{F}\Big(\frac{2}{\sqrt{C^4+4}}\,,\,a^2\Big)\,,\quad \mathrm{e}^F = 1-\frac{4\,a^4}{C^4+4}\,,\qquad 0<a<1\,,$$

$\tilde{F}(z,k)$ denotes the incomplete elliptic integral of the first kind in the Legendre form, with z being the sine of the amplitude and k the modulus;

$$f = \mathrm{e}^{-2F}\left\{\frac{(1-a^4)(3-a^4)}{3}+\frac{C^2}{4}\,Y^2+\frac{2(1-a^4)}{3\,C}\,Y+\frac{C^2}{6}\,a^4(4-a^4)\times\right.$$

$$\times[\arctan(C+1)-\arctan(C-1)]-m\left(2\frac{1-a^4}{C}+\frac{3}{2}\,C^2\,Y\right)-$$

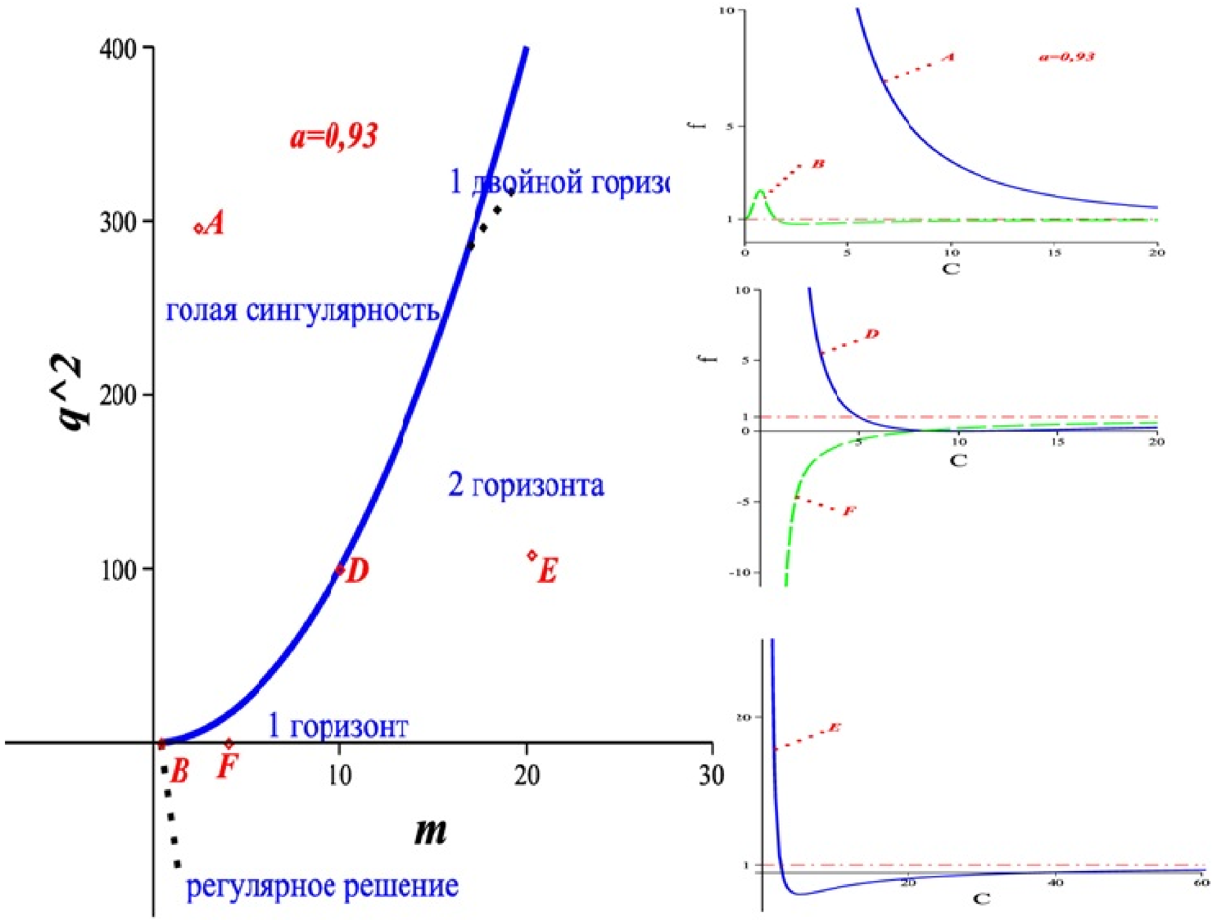
$$-\,4\,q^2\,C^2\,\mathrm{e}^{-2F}\int_C^\infty\frac{P\,\mathrm{e}^F}{C^4}dC\,,$$

$$P(C) = -\frac{Y}{2}+\frac{a^4}{2}\ln\sqrt{\frac{C^2-2C+2}{C^2+2C+2}}+\frac{\pi\,a^4}{4},$$

$$Y = \frac{a^4}{2}\left\{\ln\sqrt{\frac{C^2-2C+2}{C^2+2C+2}}-\arctan(C-1)-\arctan(C+1)+\pi\right\},$$

$$V(C) = \frac{8\,a^4\,C^3\,f\,e^{-F}}{(C^4+4)^2}+\frac{2\,(Y+C-3\,m-2q^2P)\,e^{-F}-3\,f+1}{2\,C^2}-\frac{q^2}{2\,C^4}.$$

The phase space of the family for a fixed value of parameter $a = 0.93$. The relationship between the type of solution and the characteristic function $f(C, m, a, q)$ for $a = 0.93$.



An extremal black hole

Under **extremal black hole** we mean a solution, restricted by the conditions:

$$\left\{ \begin{array}{rcl} f(C^*) & \geq & 0\,, \\ \exists\,!\,C^* > 0 : \; f(C^*) & = & 0\,; \end{array} \right.$$

$$\sim$$

$$\left\{ \begin{array}{rcl} f(C^*) & = & 0\,, \\ f^{\vee}(C^*) & = & 0\,. \end{array} \right.$$

The values of mass and charge can be found from the system of equations:

$$\left\{ \begin{array}{c} \displaystyle\int_{C^*}^{\infty} \frac{Q\,\mathrm{e}^{F}}{C^{4}} dC \; - \; 3\,m \int_{C^*}^{\infty} \frac{\mathrm{e}^{F}}{C^{4}} dC \; - \; 2\,q^2 \int_{C^*}^{\infty} \frac{P\,\mathrm{e}^{F}}{C^{4}} dC \; = \; 0\,, \\ Q(C^*) \; - \; 3\,m(C^*) \; - \; 2\,q^2 P(C^*) \; = \; 0. \end{array} \right.$$

Proposition. *For every* $C^* > 0$ *there exist unique values of variables* $q^2 > 0$, $m > 0$ *such that solution solution is an extremal black hole with a horizon radius* $C = C^*$.

Exact expressions for the mass and charge squared for an extreme black hole:

$$m = \frac{1}{3} \cdot \frac{P \int_{C^*}^{\infty} \frac{Q\,\mathrm{e}^F}{C^4} dC - Q \int_{C^*}^{\infty} \frac{P\,\mathrm{e}^F}{C^4} dC}{P \int_{C^*}^{\infty} \frac{\mathrm{e}^F}{C^4} dC - \int_{C^*}^{\infty} \frac{P\,\mathrm{e}^F}{C^4} dC},$$

$$q^2 = \frac{1}{2} \cdot \frac{Q \int_{C^*}^{\infty} \frac{\mathrm{e}^F}{C^4} dC - \int_{C^*}^{\infty} \frac{Q\,\mathrm{e}^F}{C^4} dC}{P \int_{C^*}^{\infty} \frac{\mathrm{e}^F}{C^4} dC - \int_{C^*}^{\infty} \frac{P\,\mathrm{e}^F}{C^4} dC}.$$

Proposition. *Functions* $q^2(C^*)$ *and* $m(C^*)$ *monotonically increase, where* C^* *– the radius of double event horizon.*

$\Longrightarrow$ Charge and mass increase with horizon growth.

Reissner-Nordstrom solution

$$ds^2 = \left(1 - \frac{2m}{C} + \frac{q^2}{C^2}\right) dt^2 - \frac{dC^2}{1 - \frac{2m}{C} + \frac{q^2}{C^2}} - C^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,.$$

The event horizons for the spacetime are located where the characteristic function becomes zero; that is, where

$$1 - \frac{2m}{C} + \frac{q^2}{C^2} = 0 \Leftrightarrow C = m \pm \sqrt{m^2 - q^2}\,.$$

If $m < q$ the solution is a naked singularity; if $m > q$ the solution is a black hole with two horizons: the event horizon and an internal Cauchy horizon; if $m = q$ the solution is an extreme black hole.

The mass-to-charge ratio for black holes:

$$\frac{m}{q} \geq 1\,.$$

Black hole electron:

$$m_e = 4.19 * 10^{-23},\ q = e \approx 0,085 \Rightarrow \frac{m}{q} \ll 1\,.$$

The Reissner–Nordstrom metric has a naked singularity.

The mass-to-charge ratio

Theorem. *The ratio* $\frac{m}{|q|}$ *for a black hole with scalar and electromagnetic fields is greater than* $\frac{2\sqrt{2}}{3}$, *furthermore this estimate is accurate.*

For each black hole solution there is a value $\widetilde{C^*} > C^*$:

$$Q(\widetilde{C^*}) - 3\,m - 2\,q^2\,P(\widetilde{C^*}) = 0 \implies \frac{m}{q} = \frac{1}{3}\left(\frac{Q(\widetilde{C^*})}{q} - 2qP(\widetilde{C^*})\right).$$

From the obvious estimates, it follows that

$$\frac{a}{q} + bq \geq 2\sqrt{ab} \implies \frac{m}{q} \geq \frac{2\sqrt{2}}{3}\sqrt{-P(\widetilde{C^*})Q(\widetilde{C^*})}.$$

Next, we prove that the functions

$$P(C) = -\int_C^\infty \frac{\mathrm{e}^F}{C^2}dC \text{ и } Q(C) = C + \int_C^\infty (1 - \mathrm{e}^F)dC$$

satisfy the condition $-PQ > 1$.

$$\implies \frac{m}{q} > \frac{2\sqrt{2}}{3} \approx 0,942809.$$

Let's consider the family of functions $\mathrm{e}^{F(C)}$

$$\mathrm{e}^{F(C)} = \begin{cases} \frac{1}{a^2}, & \text{если}\, C \in [0; a); \\ 1, & \text{если}\, C \geq a. \end{cases}$$

The parameters m and q^2 are chosen by the corresponding an extremal black hole with the radius of double event horizon $C^* = 1$.

$$\Longrightarrow \frac{q}{m} = 3\sqrt{a\,(a^5 + 3\,a^3 - 2\,a^2 - 3\,a + 2)} \times$$

$$\times \frac{a^2\sqrt{2\,a^4 + 4\,a^3 - 4\,a^2 - 4\,a + 3}}{4\,a^7 + 6\,a^6 - 2\,a^5 - 4\,a^4 - 5\,a^3 + 2\,a^2 + 3\,a - 1},$$

$$\Longrightarrow \lim_{a\to\infty} \frac{m}{q} = \frac{2\sqrt{2}}{3}.$$

The last equality proves that the estimate is accurate.

Conclusions:

1.) Using the inverse-problem method and symbolic computation tools, we show that the list of all possible types of space-time causal structure in the models under consideration is the same as the one for Reissner-Nordstrom.

2.) We present new result concerning the mass-to-charge ratio of black hole, namely, we obtain the infimum of the mass-to-charge ratio for both extreme and non-extreme black-hole solutions with positive kinetic energy.

References

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