Fractional stochastic field theory

Juha Honkonen, Finnish National Defence University

July 5, 2017

Outline

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Conclusion

Lévy flights and random walks

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The ordinary diffusion term is brought about by the small-scale part of the step distribution.

Long tails in waiting times

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Memory effects follow. May be described by integral operators, which give rise to fractional differentiation and integration.

Fractional derivatives

Fractional derivative of Riemann-Liouville

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Easy to handle through Laplace transform:

$$\mathcal{L}\left(\mathcal{D}_+^\alpha f\right)(s) = s^\alpha \mathcal{L} f(s)\,.$$

Fractional diffusion equation

Deform right-hand side of the diffusion equation

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The Green function of this fractional differential equation

$$\Delta_{12}^{ML}(t,\mathbf{k}) = \theta(t) E_\alpha\left[\left(-D_\sigma k^\sigma - Dk^2\right) t^\alpha\right]\,.$$

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Definition of the Mittag-Leffler function E_α by power series and integral representation:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} = \frac{1}{2\pi i}\int ds \frac{e^s s^{\alpha-1}}{s^\alpha - z}\,.$$

Fractional kinetic equation

Let $\varphi[\tilde{A}]$ be solution of the generic fractional kinetic equation

$$\frac{\partial \varphi}{\partial t} = \mathcal{D}_+^{1-\alpha} V(\varphi) = -K\mathcal{D}_+^{1-\alpha}\varphi + \mathcal{D}_+^{1-\alpha} U(\varphi) + \mathcal{D}_+^{1-\alpha}\tilde{A}\,,$$

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Note that there is no randomness yet, $\tilde{A}$ is a fixed function.

Functional representation

Perturbation expansion given by the S-matrix functional

$$G(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}^{ML'}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left[\tilde{\varphi}\mathcal{D}_{+}^{1-\alpha}U(\varphi) + \tilde{\varphi}\mathcal{D}_{+}^{1-\alpha}\tilde{A} + A\varphi\right]\Big|_{\tilde{\varphi}=\varphi=0}$$

where $\Delta_{12}^{ML'}(t,\mathbf{x};t,\mathbf{x}') \equiv 0$.

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Contraction with the attached ML-propagator saves our day (Fourier-Laplace space)

$$\Delta_{12}^{ML}(t, \mathbf{k})\mathcal{D}_{+}^{1-\alpha} \to \frac{s^{\alpha-1}}{s^{\alpha} + Dk^2 + D_{\sigma}k^{\sigma}}\, s^{1-\alpha}$$
$$= \frac{1}{s^{\alpha} + Dk^2 + D_{\sigma}k^{\sigma}} = \Delta_{12}^{\alpha}(s, \mathbf{k})\,.$$

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The final perturbation expansion contains vertices local in time only!

Change of auxiliary variable

Generating function with nonlocal interaction

$$G(A) = \exp\left(\frac{\delta}{\delta\varphi}{\Delta'}_{12}^{ML}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left[\tilde{\varphi}\mathcal{D}_{+}^{1-\alpha}U(\varphi)+\tilde{\varphi}\mathcal{D}_{+}^{1-\alpha}\tilde{A}+A\varphi\right]\Big|_{\tilde{\varphi}=\varphi=0}.$$

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Integration by parts yields

$$\tilde{\varphi}\mathcal{D}_{+}^{1-\alpha}U(\varphi) = U(\varphi)\mathcal{D}_{-}^{1-\alpha}\tilde{\varphi} = \int dt U(\varphi(t))\left(-\frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{t}^{\infty}du\frac{\tilde{\varphi}(u)}{(t-u)^{1-\alpha}}\right).$$

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In terms of the new variable $\tilde{\phi} = \mathcal{D}_{-}^{1-\alpha}\tilde{\varphi}$ the generating function

$$G(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta^{\alpha}_{12}{}'\frac{\delta}{\delta\tilde{\phi}}\right)\exp\left[\tilde{\phi}U(\varphi) + \tilde{A}\tilde{\phi} + A\varphi\right]\Big|_{\tilde{\phi}=\varphi=0}$$

contains a local interaction functional.

Langevin equation

Langevin equation is the fractional kinetic equation

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with the white-in-time Gaussian noise

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Integration of the generating functional over noise yields

$$G(A) = \exp\left(\frac{\delta}{\delta \varphi} {\Delta_{12}^{\alpha}}' \frac{\delta}{\delta \tilde{\phi}} \right) \exp\left[\tilde{\phi} U(\varphi) + \frac{1}{2} \tilde{\phi} D \tilde{\phi} + A \varphi \right] \Big|_{\tilde{\phi} = \varphi = 0}$$

De Dominicis-Janssen action

With the use of the identity

$$\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}^{\alpha}{}'\frac{\delta}{\delta\tilde{\phi}}\right)=\int\mathcal{D}\phi\,\int\mathcal{D}\tilde{\varphi}\,\exp\left[\tilde{\varphi}\left(-\mathcal{D}_{+}^{\alpha}-K\right)\phi+\phi\frac{\delta}{\delta\varphi}+\tilde{\varphi}\frac{\delta}{\delta\tilde{\phi}}\right]$$

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The usual time derivative is often generated by fluctuations, thus the generic propagator is

$$\Delta_{12}^{\alpha}(s,\mathbf{k})=\frac{1}{c_1s+c_\alpha s^\alpha+Dk^2+D_\sigma k^\sigma}\,.$$

Renormalized field theory

Renormalization requires homogeneity of the propagator, separate the fractional terms to interaction:

$$
\begin{aligned}
S_0 &= \tilde{\varphi}\left(-c_1\partial_t + D\nabla^2\right)\phi + \frac{1}{2}\tilde{\varphi}D\tilde{\varphi}\,, \\
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Large-scale relevance of local and fractional terms depends on fluctuations: renormalization produces changes in scaling dimensions of local terms.

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$$\frac{\left(c_\alpha s^\alpha + D_\sigma k^\sigma\right)^n}{\left(c_1 s + D k^2\right)^{n+1}} \,.$$

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Although popular, this approach is dubious.

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The MS ray scheme does not share this property.

Renormalization in the ray scheme

A divergent (sub)graph gives rise to a factor of the form (n, m, l – integers)

$$\frac{1}{n\varepsilon + m(2-\sigma) + 2l(1-\alpha)} = \frac{1}{\varepsilon}\,\frac{1}{n + m\zeta + 2l\xi}\,.$$

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Result: coefficient functions of the renormalization group are meromorphic functions in ε, $2-\sigma$ and $1-\alpha$, not analytic as they should be!

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Result: coefficient functions of the renormalization group are meromorphic functions in ε, $2-\sigma$ and $1-\alpha$, not analytic as they should be!

The MS ray scheme is at least dubious.

Renormalization in the ray scheme

A divergent (sub)graph gives rise to a factor of the form (n, m, l – integers)

$$\frac{1}{n\varepsilon + m(2-\sigma) + 2l(1-\alpha)} = \frac{1}{\varepsilon}\,\frac{1}{n + m\zeta + 2l\xi}\,.$$

Contribution of a graph to renormalization constant: product of these multiplied by a function analytic in ε, $2-\sigma$ and $1-\alpha$ at the origin.

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Result: coefficient functions of the renormalization group are meromorphic functions in ε, $2-\sigma$ and $1-\alpha$, not analytic as they should be!

The MS ray scheme is at least dubious. Use the the normalization point scheme instead.

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- ☐ Fractional differential operators give rise to multi-parameter regularization.
- ☐ MS ray scheme is not consistent beyond one-loop order.
- ☐ In multiloop calculations dimensional-analytic regularization with normalization-point subtractions is the safe (and technically challenging) way to proceed.