SYMBOLIC-NUMERICAL MODELING OF THE INFLUENCE OF DAMPING MOMENTS ON SATELLITE DYNAMICS

Gutnik S.A.

Department of Informatics and Computational Mathematics

Moscow Institute of Physics and Technology (University)

Sarychev V.A.

Keldysh Institute of Applied Mathematics, RAS

The International Conference “Mathematical Modeling and Computational Physics, 2017” (MMCP2017) 3 - 7 July 2017, Dubna, Russia

1. Equations of motion

The equations of the satellite's attitude motion under the influence of gravitational and active damping torques in a circular orbit take the form:

$$\begin{aligned}
&Ap_1' + (C-B)q_1r_1 - 3\omega_0^2(C-B)a_{32}a_{33} + \bar{k}_1 p_1 = 0,\\
&Bq_1' + (A-C)r_1p_1 - 3\omega_0^2(A-C)a_{33}a_{31} + \bar{k}_2 q_1 = 0,\\
&Cr_1' + (B-A)p_1q_1 - 3\omega_0^2(B-A)a_{31}a_{32} + \bar{k}_3 r_1 = 0;
\end{aligned} \tag{1}$$

$$\begin{aligned}
&p_1 = (\alpha' + \omega_0)a_{21} + \gamma',\\
&q_1 = (\alpha' + \omega_0)a_{22} + \beta' \sin\gamma,\\
&r_1 = (\alpha' + \omega_0)a_{23} + \beta' \cos\gamma.
\end{aligned} \tag{2}$$

Here ***A, B, C*** are the principal central moments of inertia of the satellite; p_1, q_1, r_1 – are the projections of the satellite's angular velocity on the axes *Ox, Oy, Oz; α, β, γ - are* the angles of pitch, yaw and roll; a_{ij} - the direction cosines of the axis *Ox*, *Oy*, *Oz* in the orbital reference frame. The projections of integral vector of active damping moments on the axis of the body reference frame: $M_x = \bar{k}_1 p_1$, $M_y = \bar{k}_2 q_1$, $M_z = \bar{k}_3 r_1$, $\bar{k}_1, \bar{k}_2, \bar{k}_3$ - are the damping coefficients ; ω_0 - is the angular velocity of the orbital motion of the satellite's center of mass.

2. Equations of motion

Introducing dimensionless parameters $\theta_A = A/B,\ \theta_C = C/B,\ p = p_1/\omega_0,\ q = q_1/\omega_0,$ $r = r_1/\omega_0,\ \tilde{k}_1 = \bar{k}_1/\omega_0 B,\ \tilde{k}_2 = \bar{k}_2/\omega_0 B,\ \tilde{k}_3 = \bar{k}_3/\omega_0 B,\ \ \tau = \omega_0 t$ the system (1), (2) takes the form

$$
\begin{aligned}
&\theta_A \dot{p} + (\theta_C - 1)qr - 3(\theta_C - 1)a_{32}a_{33} + \tilde{k}_1 p = 0,\\
&\dot{q} + (\theta_A - \theta_C)rp - 3(\theta_A - \theta_C)a_{33}a_{31} + \tilde{k}_2 q = 0,\\
&\theta_c \dot{r} + (1 - \theta_A)pq - 3(1 - \theta_A)a_{31}a_{32} + \tilde{k}_3 r_1 = 0;
\end{aligned}
\qquad (3)
$$

$$
\begin{aligned}
&p = (\dot{\alpha} + 1)a_{21} + \dot{\gamma},\\
&q = (\dot{\alpha} + 1)a_{22} + \dot{\beta}\sin\gamma,\\
&r = (\dot{\alpha} + 1)a_{23} + \dot{\beta}\cos\gamma.
\end{aligned}
\qquad (4)
$$

Putting in (3) - (4) $\alpha = \alpha_0 = \mathrm{const},\ \beta = \beta_0 = \mathrm{const},\ \gamma = \gamma_0 = \mathrm{const}$ we obtain the equations

$$
\begin{aligned}
&a_{22}a_{23} - 3a_{32}a_{33} + k_1 a_{21} = 0, && a_{21}^2 + a_{22}^2 + a_{23}^2 = 1,\\
&a_{23}a_{21} - 3a_{33}a_{31} + k_2 a_{22} = 0, && a_{31}^2 + a_{32}^2 + a_{33}^2 = 1,\\
&a_{21}a_{22} - 3a_{31}a_{32} + k_3 a_{23} = 0, && a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0.
\end{aligned}
\qquad (5)
$$

which specifies all equilibrium orientations of a satellite in the orbital coordinate system.

Here $\quad k_1 = \tilde{k}_1 / (\theta_C - 1),\ k_2 = \tilde{k}_2 / (\theta_A - \theta_C),\ k_3 = \tilde{k}_3 / (1 - \theta_A)\,.$

3. Equilibrium orientations of a satellite

For the system of equations (5) we state the following problem: for given values of $k_1,\ k_2,\ k_3$ determine all nine directional cosines, i.e., all equilibrium orientations of a satellite in the orbital coordinate system.

To find the solutions of the algebraic system (5), we constructed the Groebner basis of the system of six second-order polynomials with six variables a_{ij} (i=2,3; j=1,2,3), with respect to the lexicographic ordering of variables by using option ***plex***: G:=map(factor,Groebner[Basis]([F, plex(a31, ... a22))).
Here we write down the polynomial in the Groebner basis that depends only on one variable a_{22}. in the form

$$a_{22}(a_{22}^2-1)=0. \tag{6}$$

The polynomials from the Groebner basis that depends on variables a_{21}, a_{23} have the similar forms ($a_{21}(a_{21}^2-1)=0;\ \ a_{23}(a_{23}^2-1)=0$).

The polynomial from the Groebner basis that depends on variables a_{33} is:

$$a_{33}(9a_{33}^4-9a_{33}^2+k_2^2)(9a_{33}^4-9a_{33}^2+k_1^2)=0. \tag{7}$$

3. Equilibrium orientations of a satellite

To determine the equilibria it is required to consider separately the following three cases: a_{22}=1, a_{22}=-1 and a_{22}=0.

In the first case, when $a_{22}=1\ (a_{21}=a_{23}=0)$, we obtain

$$9a_{33}^4-9a_{33}^2+k_2^2=0. \tag{8}$$

1-2. As a result, from the first and second conditions we obtain the following equilibrium solutions ($k_2^2\le 9/4$)

$$a_{22}=\pm 1,\ a_{21}=a_{23}=0;\ \ a_{32}=0,\ a_{31}^2=\frac{3\mp\sqrt{9-4k_2^2}}{6},\ a_{33}^2=\frac{3\pm\sqrt{9-4k_2^2}}{6}. \tag{9}$$

3. In the case $a_{22}=0$ there are two sets of equilibrium solutions:

$$a_{33}^2=1-\frac{k_1k_3}{3},\ a_{32}^2=\frac{k_1k_3}{3},\ a_{31}=0,\ a_{21}^2=1,\ a_{22}=0,\ a_{23}=0.\ (k_1=\frac{3k_3}{(k_3^2+1)}). \tag{10}$$

$$a_{31}^2=1-\frac{k_1k_3}{3},\ a_{32}^2=\frac{k_1k_3}{3},\ a_{33}=0,\ a_{21}=0,\ a_{22}=0,\ a_{23}=1.\ (k_3=\frac{3k_1}{(k_1^2+1)}). \tag{11}$$

Similarly, we can obtain other solutions (5) that are obtained from the conditions:

$$a_{21}=\pm 1,\ a_{21}=0;\quad a_{23}=\pm 1,\ \ a_{23}=0.$$

4. Conditions of asymptotic stability

The linearized system of equations of motion (3), (4) takes the following form:

$$\theta_A \ddot{\bar{\alpha}} \sin\beta_0 + [2(\theta_C - 1)a_{22}a_{23} + k_1 a_{21}]\dot{\bar{\alpha}} + 3(\theta_C - 1)(a_{12}a_{33} +$$
$$+ a_{13}a_{32})\bar{\alpha} + \cos\beta_0[(\theta_A + \theta_C - 1) - 2(\theta_C - 1)\sin^2\gamma_0]\dot{\bar{\beta}} +$$
$$+ \cos\beta_0\{(\theta_C - 1)[(1 + 3\sin^2\alpha_0)\sin\beta_0 \sin 2\gamma_0 - \frac{3}{2}\sin 2\alpha_0 \cos 2\gamma_0] +$$
$$+ k_1\}\bar{\beta} + \theta_A \ddot{\bar{\gamma}} + k_1 \dot{\bar{\gamma}} + (\theta_C - 1)[(a_{23}^2 - a_{22}^2) - 3(a_{33}^2 - a_{32}^2)]\bar{\gamma} = 0,$$

$$\ddot{\bar{\alpha}} a_{22} + [2(\theta_A - \theta_C)a_{21}a_{23} + k_2 a_{22}]\dot{\bar{\alpha}} + 3(\theta_A - \theta_C)(a_{13}a_{31} +$$
$$+ a_{11}a_{33})\bar{\alpha} + \ddot{\bar{\beta}} \sin\gamma_0 + [(\theta_A - \theta_C - 1)\sin\beta_0 \cos\gamma_0 + k_2 \sin\gamma_0]\dot{\bar{\beta}} \ \ldots$$
$$\theta_C \ddot{\bar{\alpha}} a_{23} + \cos\beta_0[2(1 - \theta_A)\sin\beta_0 \cos\gamma_0 - k_3 \sin\gamma_0]\dot{\bar{\alpha}} + 3(1 - \theta_A)\times$$
$$\times(a_{11}a_{32} + a_{12}a_{31})\bar{\alpha} + \theta_C \ddot{\bar{\beta}} \cos\gamma_0 + [(\theta_C - \theta_A + 1)\sin\beta_0 \sin\gamma_0 +$$
$$+ k_3 \cos\gamma_0]\dot{\bar{\beta}} + \{(1 - \theta_A)[(1 + 3\sin^2\alpha_0)\cos 2\beta_0 \cos\gamma_0 -$$
$$- \frac{3}{2}\sin 2\alpha_0 \sin\beta_0 \sin\gamma_0] + k_3 \sin\beta_0 \sin\gamma_0\}\bar{\beta} - (\theta_A + \theta_C - 1)\dot{\bar{\gamma}} a_{22} = 0$$

4. Conditions of asymptotic stability

Consider small oscillations of the satellite in the vicinity of the equilibrium solution

$$\alpha_0 = Arc\sin\left(\left(\frac{3-\sqrt{9-4k^2}}{6}\right)^{1/2}\right),\ \beta_0 = \gamma_0 = 0. \qquad (12)$$

For solution (12) In special case, when $k_1=k_2=k_3=k$ linearized equations take the form

$$\begin{aligned}
&\ddot{\bar{\alpha}} + k\dot{\bar{\alpha}} + 3(\theta_A - \theta_C)(\cos^2\alpha_0 - \sin^2\alpha_0)\bar{\alpha} = 0,\\
&\theta_C\ddot{\bar{\beta}} + k\dot{\bar{\beta}} - (\theta_A + \theta_C - 1)\dot{\bar{\gamma}} + (1-\theta_A)(1+3\sin^2\alpha_0)\bar{\beta} +\\
&+[3(1-\theta_A)\sin\alpha_0\cos\alpha_0 - k]\bar{\gamma} = 0,\\
&\theta_A\ddot{\bar{\gamma}} + (\theta_A + \theta_C - 1)\dot{\bar{\beta}} + k\dot{\bar{\gamma}} + [3(1-\theta_C)\sin\alpha_0\cos\alpha_0 + k]\bar{\beta} +\\
&+(1-\theta_C)(3\cos^2\alpha_0 + 1)\bar{\gamma} = 0.
\end{aligned} \qquad (13)$$

The characteristic equation of system (13) takes the form

$$[\lambda^2 + k\lambda + (\theta_A - \theta_C)\sqrt{9-4k^2}](A_0\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4) = 0, \qquad (14)$$

4. Conditions of asymptotic stability

In (14)

$$A_0 = \theta_A \theta_C, \quad A_1 = k(\theta_C + \theta_A),$$

$$A_2 = k^2 + (\theta_A + \theta_C - 1)^2 + \theta_A(1-\theta_A)\frac{5-\sqrt{9-4k^2}}{2} + \theta_C(1-\theta_C)\frac{5+\sqrt{9-4k^2}}{2},$$

$$A_3 = k[(\theta_A + \theta_C - 1)(\theta_A - \theta_C + 2) + (1-\theta_C)\frac{5+\sqrt{9-4k^2}}{2} + \qquad (15)$$

$$+(1-\theta_A)\frac{5-\sqrt{9-4k^2}}{2}], \ A_4 = (1-\theta_A)(1-\theta_C)(3+k^2) + (\theta_A + \theta_C - 1)k^2.$$

The necessary and sufficient conditions for asymptotic stability (Routh-Hurwitz criterion) of the equilibrium solution (12) take the following form:

$$k > 0, \quad \theta_A - \theta_C > 0,$$

$$\Delta_1 = A_1 = k(\theta_A + \theta_C) > 0,$$

$$\Delta_2 = A_1 A_2 - A_0 A_3 > 0, \qquad (16)$$

$$\Delta_3 = A_1 A_2 A_3 - A_0 A_3^2 - A_1^2 A_4 > 0,$$

$$\Delta_4 = \Delta_3 A_4 > 0, \quad A_4 > 0.$$

The triangle inequalities should also be satisfied

$$1+\theta_A \ge \theta_C, \ 1+\theta_C \ge \theta_A, \ \theta_A + \theta_C \ge 1. \qquad (17)$$

Example where necessary and sufficient conditions of stability (16) and (17) hold

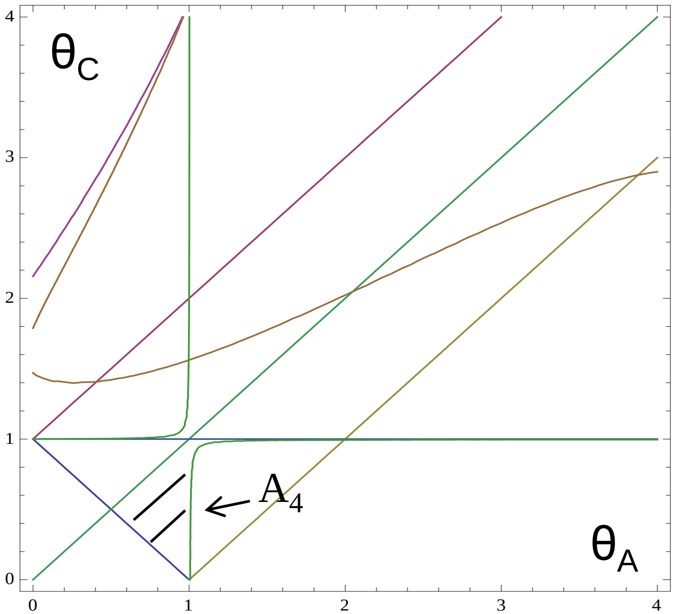

Fig.1 *k=0.1*

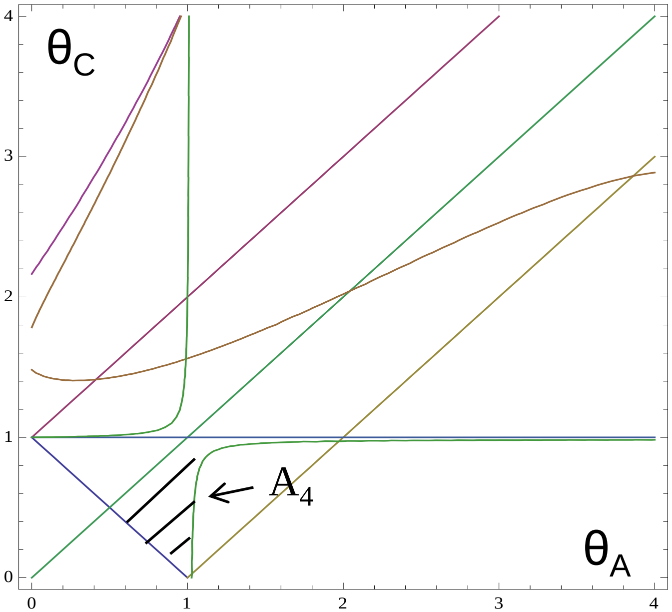

Fig.2 *k=0.2*

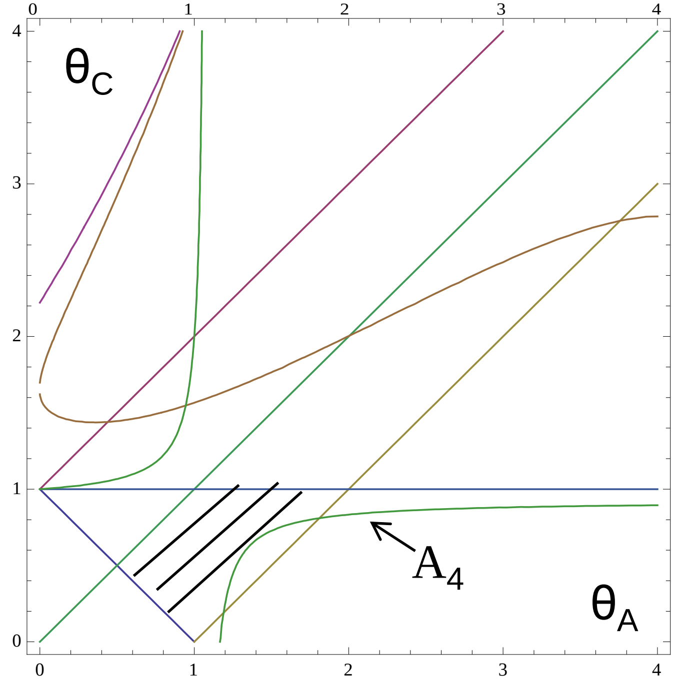

Fig.3 *k=0.5*

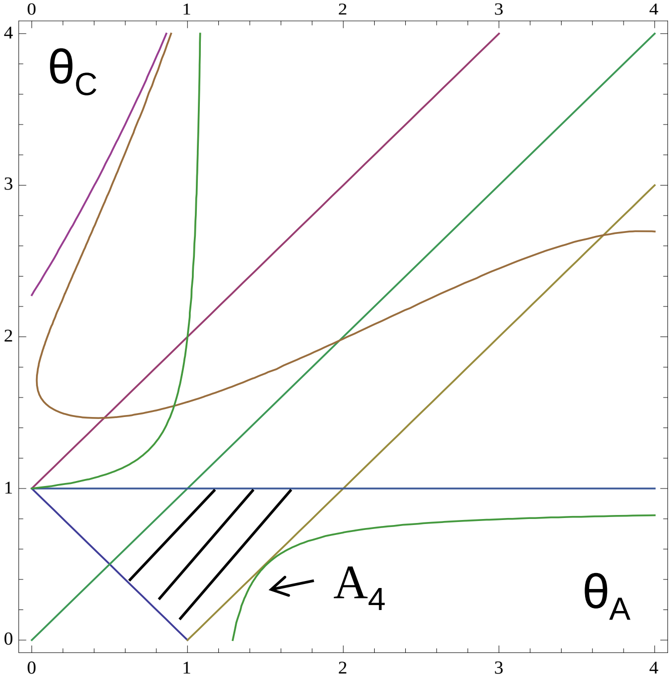

Fig.4 *k=0.65*

5.Analysis of transition decay processes

The numerical integration of the system (3), (4) has been done in special case $k_1 = k_2 = k_3 = k$

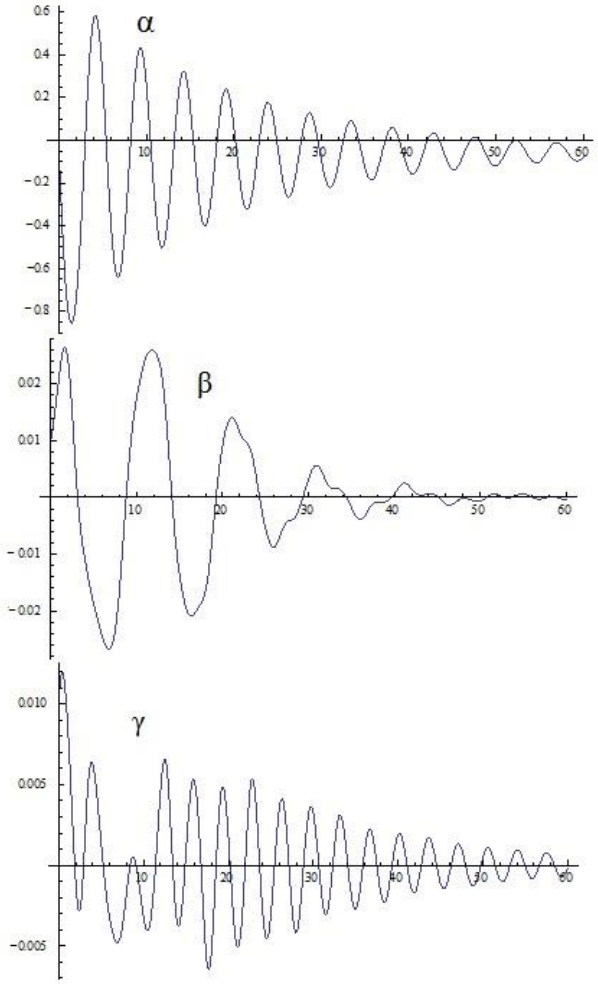

Fig.5 θ_A=0.9, θ_C=0.3, k=0.1, α_0=-0.033

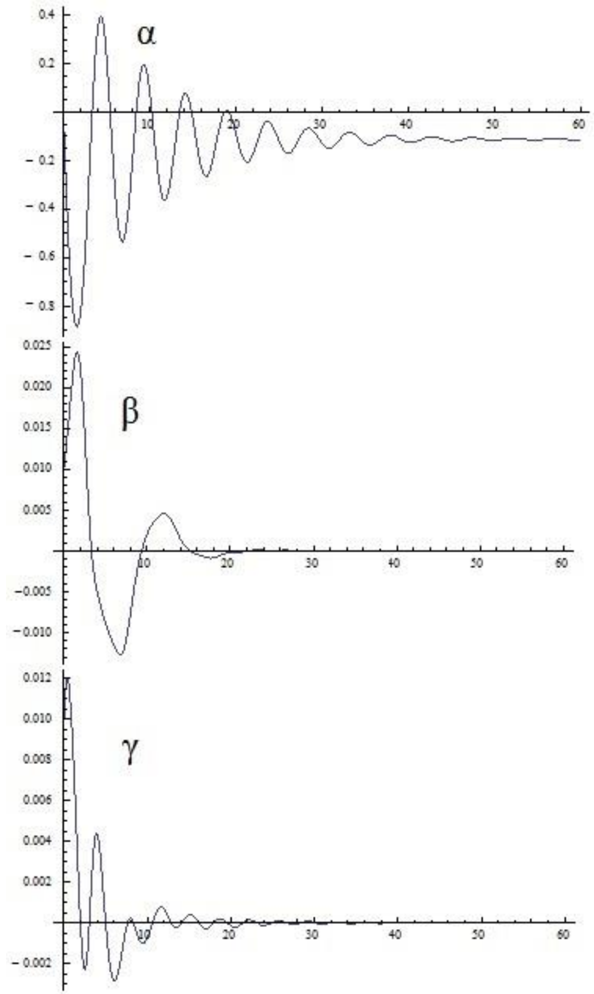

Fig.6 θ_A=0.9, θ_C=0.3, k=0.2, α_0=-0.067

5.Analysis of transition decay processes

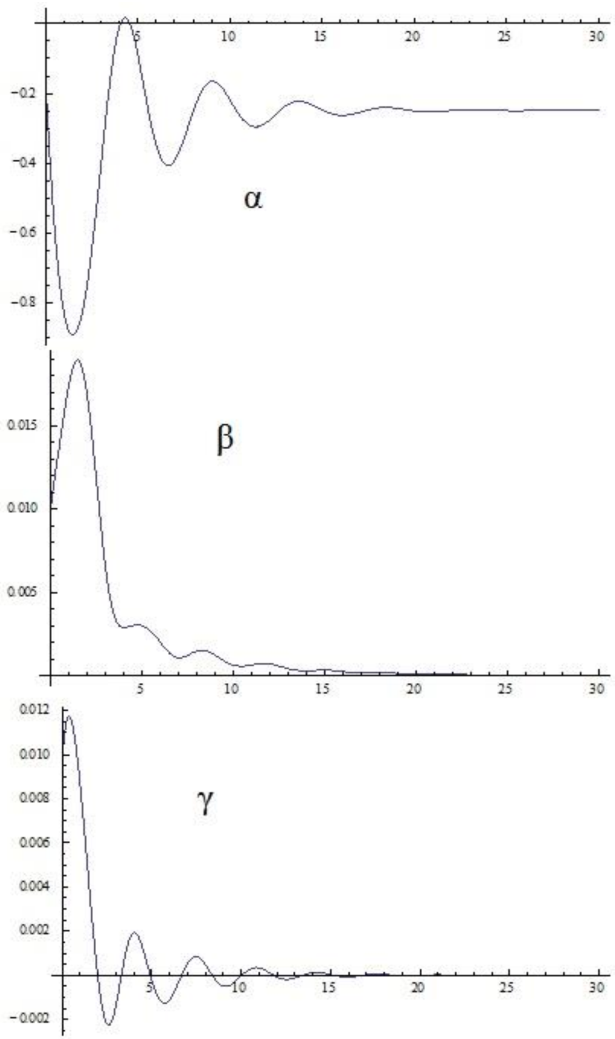

Fig.7 θ_A=0.9, θ_C=0.2, k=0.5, α_0=-0.17

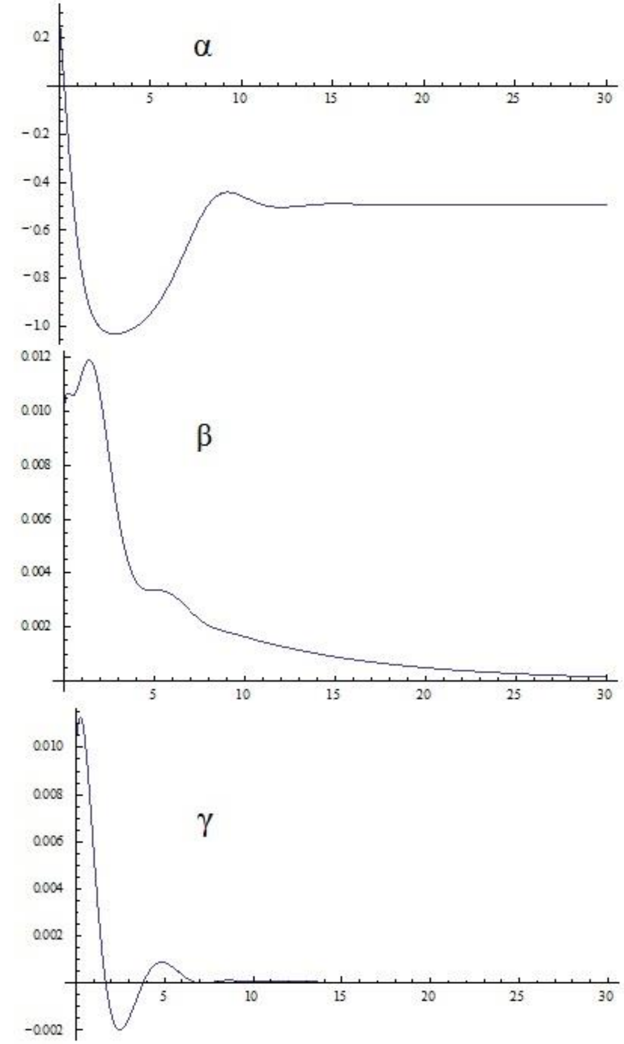

Fig.8 θ_A=0.9, θ_C=0.1, k=1.0, α_0=-0.365

6. Conclusion

- The attitude motion of the satellite relative to the center of mass in a circular orbit due to gravitational and active damping torques have been analyzed
- Necessary and sufficient conditions for asymptotic stability of the equilibrium orientations were obtained with the help of the Routh-Hurwitz criterion
- The evolution of domains where necessary and sufficient conditions for asymptotic stability of the equilibrium orientations are satisfied was investigated analytically in the plane of two inertial parameters at a different values of damping parameter k
- The transition decay processes of spatial oscillations of the satellite have been investigated numerically at a different values of inertial parameters and damping parameter k

7. References

- 1. *Sarychev V.A.* Problems of orientation of satellites. Itogi Nauki I Tekhniki. Ser. Space Research. VINITI, Moscow 11(1978)
- 2. Beletsky V.V. Attitude Motion of Satellite in Gravitational Field. Moscow: MGU Press, 1975.
- 3. Gutnik S.A., Sarychev, V.A.: A Symbolic Study of the Satellite Dynamics Subject to Damping Torques (submitted in CASC 2017)
- 4*. Buchberger B.* Theoretical basis for the reduction of polynomials to canonical forms // SIGSAM Bulletin. 1976. P. 19-29.
- 5*.Char B.W., Geddes K.O., Gonnet G.H., Monagan M.B., Watt S.M.* Maple Reference Manual // Watcom Publications Limitid. Waterloo. Canada. 1992.