On generation of random ensembles of mixed states for quantum bipartite systems

Ilia Rogojin [1] and Arsen Khvedelidze [2,3]

[1] Laboratory of Information Technologies, JINR, Dubna, Russia
[2] Institute of Quantum Physics and Engineering Technologies, GTU, Tbilisi, Georgia
[3] A. Razmadze Mathematical Institute, TSU, Tbilisi, Georgia

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Plan

1. THE GEOMETRY OF STATE SPACE
   - A generic 2-qubit system
2. MODELING STATE SPACE
   - Rank 3 states for 2-qubit system
   - Rank 2 states for 2-qubit system
   - Rank 1 states for 2-qubit system
3. SEPARABILITY vs. ENTANGLEMENT
   - Definition and criterion of separability
4. Computing separability
   - Generating random density matrices
   - Results of computation

State space

- Observables – Hermitian operators from the set of linear operators on a complex Hilbert space $\mathcal{H}$, ($\dim\mathcal{H} = N$) .
- State – the density operator ϱ , a normalized linear operator on $\mathcal{H}$ satisfying conditions:
  1. self-adjoint: $\varrho = \varrho^+$,
  2. positive semi-definite: $\varrho \geq 0$,
  3. unit trace: $\mathrm{Tr}\varrho = 1$,
- State space – the set $\mathfrak{P}_+$ of states .

State space for binary composites

PRINCIPLE OF SUPERPOSITION

- The Hilbert space $\mathcal{H}_{A\otimes B}$ for bipartite system composed from A and B subsystems is given by the tensor product of their Hilbert spaces $\mathcal{H}_A^{d_A}$ and $\mathcal{H}_B^{d_B}$:

$$\mathcal{H}_{A\otimes B} \sim \mathcal{H}_A^{d_A} \otimes \mathcal{H}_B^{d_B} \,,$$

$d_A = \dim \mathcal{H}_A^{d_A}\,, \quad d_B = \dim \mathcal{H}_B^{d_B}\,.$

- The density matrix of joint system ϱ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$

Partial trace and reduced density matrices

Information on subsystems

- Information on subsystems of the $\mathcal{H}_{A\otimes B}$ is accumulated in the reduced density matrices ϱ_A and ϱ_A:
- The partial trace of ϱ with respect to the subsystem B, defines the reduced matrix ϱ_A

$$\varrho_A = \mathrm{Tr}_B\,(\varrho)\ ,$$

- Similarly, the reduced matrix ϱ_B is given by "partial tracing" the subsystem A

$$\varrho_B = \mathrm{Tr}_A\,(\varrho)\ ,$$

A generic 2-qubit state

- A generic 2-qubit density matrices admits the form

$$\varrho = \frac{ZZ^\dagger}{\mathrm{Tr}\,(ZZ^\dagger)}$$

where Z is an arbitrary complex 4×4 matrix

- Emphasising a composite structure of 2-qubit, the so-called Fano basis is often used

$$\varrho = \frac{1}{4}\left(\mathrm{I}_2 \otimes \mathrm{I}_2 + \vec{a}\cdot\vec{\sigma} \otimes \mathrm{I}_2 + \mathrm{I}_2 \otimes \vec{b}\cdot\vec{\sigma} + c_{ij}\,\sigma_i \otimes \sigma_j\right)\,,$$

where $\vec{a}$ and $\vec{b}$ - are the Bloch vectors of the individual qubits and c_{ij} - correlation matrix

Block form of a generic 2-qubit state

- A generic 4×4 density matrix can be represented in the block form with 2×2 matrices A, B, C and D:

$$\varrho = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

- The corresponding reduced matrix ϱ_A reads

$$\varrho_A = \begin{pmatrix} \operatorname{tr}(A) & \operatorname{tr}(B) \\ \operatorname{tr}(C) & \operatorname{tr}(D) \end{pmatrix}$$

2-qubit rank 3 density matrices

- An arbitrary 4×4 complex rank 3 matrix Z can be written (up to permutation of entries by P_Z and Q_Z as

$$Z = P_Z \left(\begin{array}{ccc|c} & & & z_1 \\ & A & & z_2 \\ & & & z_3 \\ \hline y_1 & y_2 & y_3 & D \end{array}\right) Q_Z \,,$$

where the complex number D is given by formula

$$D = Y\, A^{-1} Z,$$

for any regular matrix A, 3-row $Y = (y_1, y_2, y_3)$ and 3-column $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$,

2-qubit rank 2 density matrices

- Any 4×4 complex rank 2 matrix can be written

$$Z = P_Z \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & b_{11} & b_{12} \\ & & b_{21} & b_{22} \\ \hline c_{11} & c_{12} & \multicolumn{2}{c}{\multirow{2}{*}{D}} \\ c_{21} & c_{22} & & \end{array} \right) Q_Z$$

the 2×2 complex matrix $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$ is given by product of 2×2 matrices C, A^{-1} and B,

$$D = C\, A^{-1} B$$

2-qubit rank 1 density matrices

- An arbitrary 4×4 complex rank 1 matrix Z can be written (up to permutation of entries by P_Z and Q_Z) as

$$Z = P_Z \left(\begin{array}{c|ccc} a & y_1 & y_2 & y_3 \\ \hline z_1 & & & \\ z_2 & & D & \\ z_3 & & & \end{array} \right) Q_Z \,,$$

where the 3×3 matrix D is given by formula

$$D = \frac{1}{a} \begin{pmatrix} z_1 y_1 & z_1 y_2 & z_1 y_3 \\ z_2 y_1 & z_2 y_2 & z_2 y_3 \\ z_3 y_1 & z_3 y_2 & z_3 y_3 \end{pmatrix} .$$

Separable & Entangled density matrices

- A bipartite system is separable if

$$\varrho = \sum_k p_k \varrho_A^k \otimes \varrho_B^k, \qquad \sum_k p_k = 1.$$

- Otherwise the state is entangled .

The separability definition is implicit !

Peres–Horodecki separability criterion

Introduce the partial transposition operation:

$$\varrho^{T_B} = I \otimes T\varrho\,, \qquad T - \text{transposition operator}$$

For the block matrix ϱ the partial transposition reads,

$$\varrho = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \qquad \varrho^{T_B} = \left(\begin{array}{c|c} A^T & B^T \\ \hline C^T & D^T \end{array}\right).$$

- The Peres–Horodecki separability criterion:

A binary $2 \otimes 2$ or $2 \otimes 3$ system is in a separable state if and only if the partial transposition of its density matrix gives again a positive-semidefinite operator.

Separability probability

- Inspired by the theory of geometric probability one can define the separability probability as

$$\mathcal{P}_{\text{sep}} = \frac{\text{Vol}_{\text{ Separable states}}}{\text{Vol}_{\text{ All states}}}$$

- $\text{Vol}_{\text{All states}} = \int_{\mathfrak{P}_+} \mathrm{d}\mu$, where $\mathfrak{P}_+$ is convex body of all states
- $\text{Vol}_{\text{ Separable states}} = \int_{\mathfrak{S}} \mathrm{d}\mu$, where $\mathfrak{S} \in \mathfrak{P}_+$ is convex body of separable states
- The measure $\mathrm{d}\mu$ is determined by the Riemannian distance on $\mathfrak{P}_+$.

Ingredients: Convex bodies and Distances

For the evaluation of the separability probability we need:

- Determine the convex bodies $\mathfrak{P}_+$ and $\mathfrak{S} \in \mathfrak{P}_+$;
- Introduce the Riemannian distance on $\mathfrak{P}_+$.

The bodies $\mathfrak{P}_+$ and $\mathfrak{P}_+$

$\mathfrak{S}$ and $\mathfrak{P}_+$ are semi-algebraic varieties given by the polynomial inequalities in elements of the density matrices.

The Riemannian distances used in our computations

- The Hilbert-Schmidt distance: $D_{\mathrm{HS}} = \sqrt{\mathrm{tr}(\varrho_1 - \varrho_2)^2}$,
- The Bures distance: $D_{\mathrm{B}} = \sqrt{2(1 - \mathrm{tr}(\sqrt{\varrho_1^{1/2}\varrho_2\varrho_1^{1/2}}))}$

The method of computation

ALGORITHM for generation of density matrices

- Generate matrix G from Ginibre ensemble, i.e., the matrices whose entries real and imaginary parts are independent normal random variables;
- Write down the matrix $\varrho_{\mathrm{HS}} = \frac{\mathrm{GG}^+}{\mathrm{tr\,GG}^+}\,.$

ϱ_{HS} is the Hilbert-Schmidt matrix.

- Test ϱ_{HS} on Peres–Horodecki P-H criterion
- $\mathcal{P}_{\mathrm{sep}} = \dfrac{\text{Number of positively PH-tested matrices}}{\text{Total number of matrices}}\,.$

Separability probabilities for generic states

System	Separable	Entangled
HS-metric		
$2 \otimes 2$	0.2424	0.7576
$2 \otimes 3$	0.0270	0.9730
Bure metric		
$2 \otimes 2$	0.0733	0.9267
$2 \otimes 3$	0.0014	0.9986

Separability probabilities for rank 3, 2, and 1

- For rank 3 states: $\mathcal{P}_{sep}^{r=3} = 0.1652$
- For rank 2 states: $\mathcal{P}_{sep}^{r=2} = 0$
- For rank 1 states: $\mathcal{P}_{sep}^{r=1} = 0$

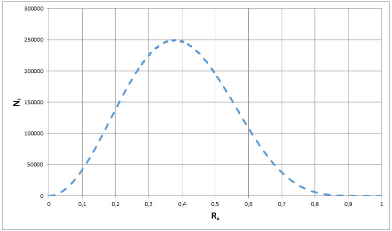
The latter two results are in accordance with the known statement:

If $\mathrm{rank}(\varrho) < d_A = \mathrm{rank}(\varrho_A)$, then ϱ is not separable.

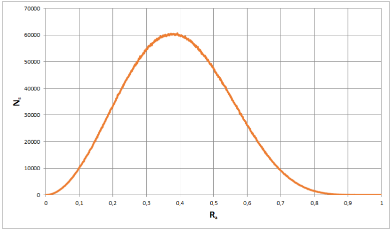
See e.g., M. Ruskai and E. M. Werner, Bipartite states of low rank are almost surely entangled, J.Phys.A.Math.Theo, 42 (2009) 095303

Distribution of Separability of H-S ensemble

The probabilistic characteristics of 2-qubit H-S ensemble



Number of the total density matrices in the Hilbert-Schmidt qubit-qubit ensemble as function of qubit Bloch vector



Number of the separable density matrices in the Hilbert-Schmidt qubit-qubit ensemble as function of qubit Bloch vector

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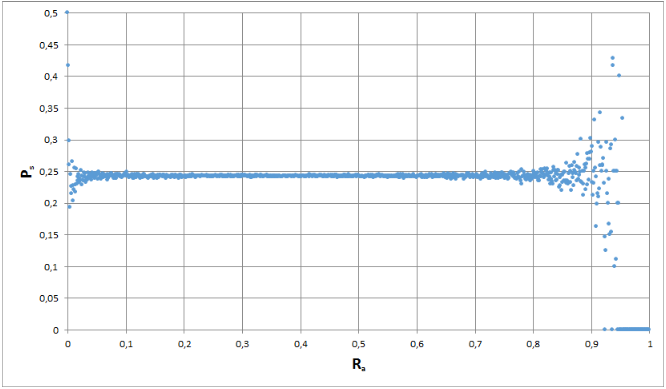
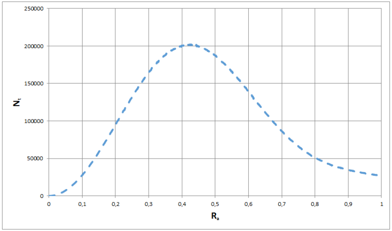


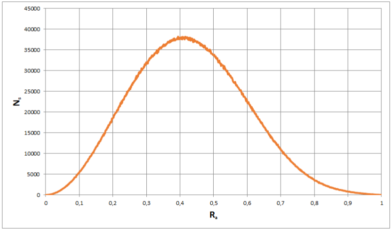
The separability probability in the Hilbert-Schmidt qubit-qubit ensemble as function of first qubit Bloch vector.

Distribution of Separability of H-S ensemble

The probabilistic characteristics of rank 3 2-qubit states in Hilbert-Schmidt ensemble



Number of the total density matrices in the Hilbert-Schmidt qubit-qubit rank 3 ensemble as function of qubit Bloch vector



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Distribution of Separability of H-S ensemble

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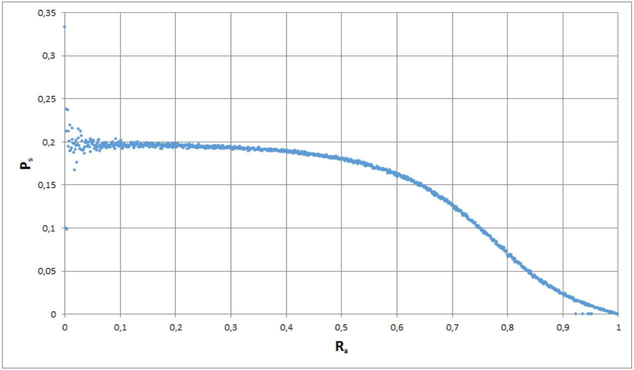
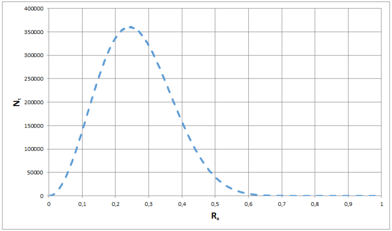


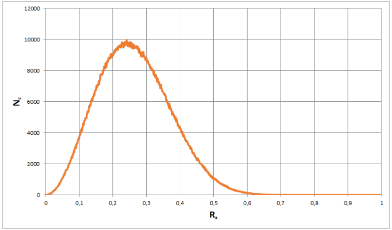
The separability probability for the rank 3 qubit-qubit states in Hilbert-Schmidt ensemble as function of first qubit Bloch vector.

Distribution of Separability of H-S ensemble

The probabilistic characteristics of qubit-qutrit Hilbert-Schmidt ensemble



Number of the total density matrices in the Hilbert-Schmidt qubit-qutrit ensemble as function of qubit Bloch vector



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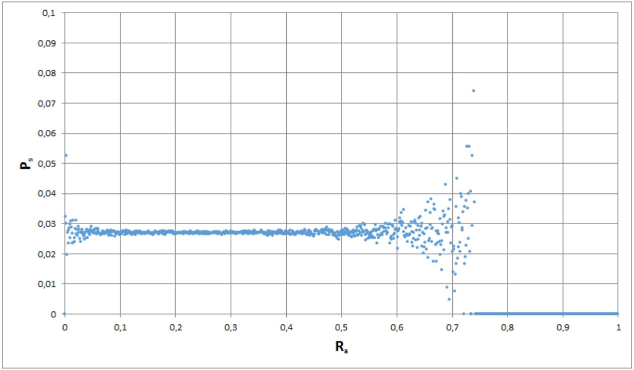
Distribution of Separability of H-S ensemble

The probabilistic characteristics of qubit-qutrit Hilbert-Schmidt ensemble



The separability probability in the Hilbert-Schmidt qubit-qutrit ensemble as function of qubit Bloch vector.

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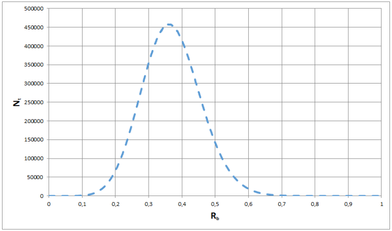
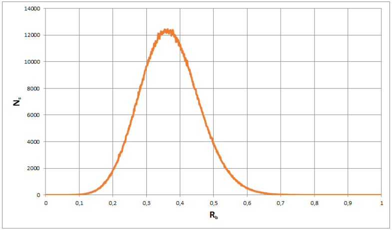


Number of the total density matrices in the Hilbert-Schmidt qubit-qutrit ensemble as function of qutrit Bloch vector



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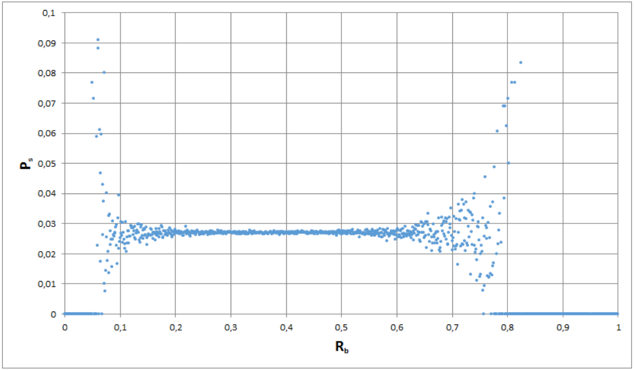
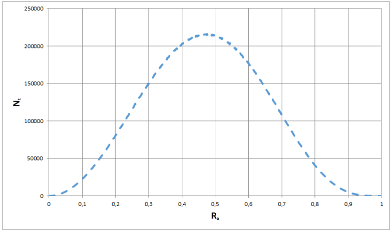


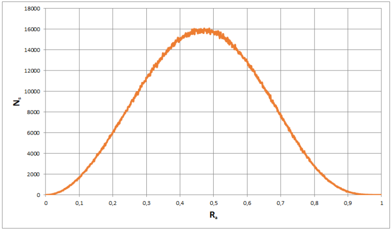
The separability probability in the Hilbert-Schmidt qubit-qutrit ensemble as function of qutrit Bloch vector.

Distribution of Separability of Bures ensemble

The probabilistic characteristics of 2-qubit Bures ensemble



Number of the total density matrices in the Bures qubit-qubit ensemble as function of qubit Bloch vector



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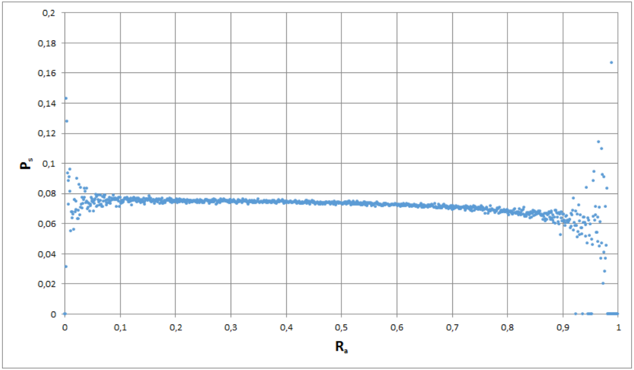

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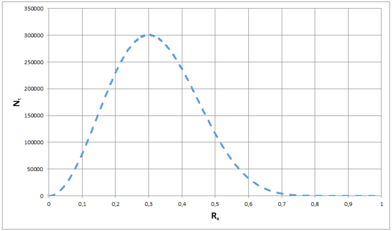
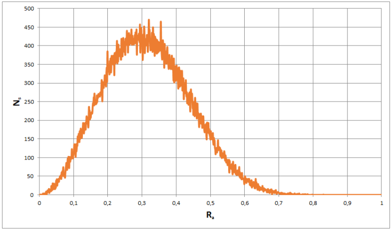


Number of the total density matrices in the Bure qubit-qutrit ensemble as function of qubit Bloch vector



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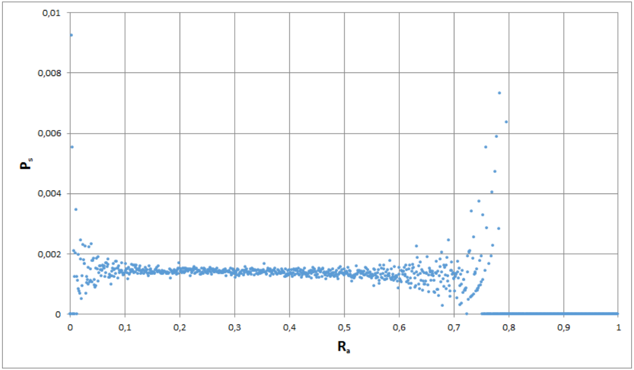

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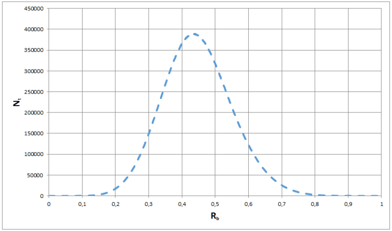

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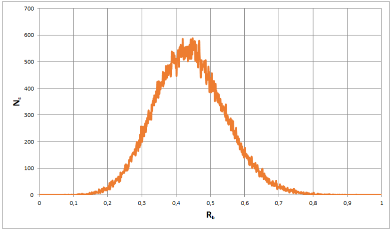

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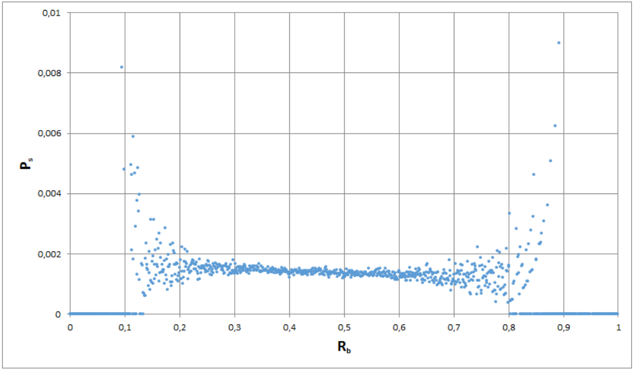

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