Near-perfect matchings on cylinders $C_m \times P_n$ of odd order

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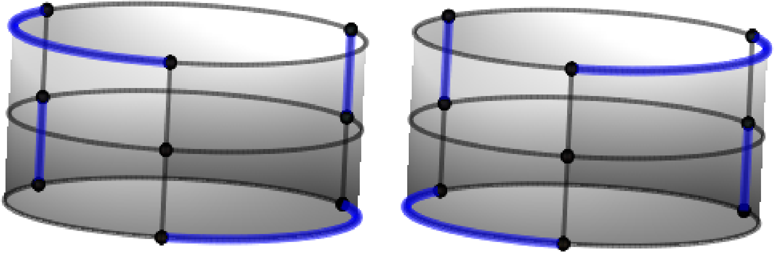
What is a near-perfect matching?

A near-perfect matching is one in which exactly one vertex is unmatched. This can only occur when the graph has an odd number of vertices, and such a matching must be maximum.

A vertex of the graph, which is not saturated by a matching, will be called vacancy.

Example of near-perfect matching

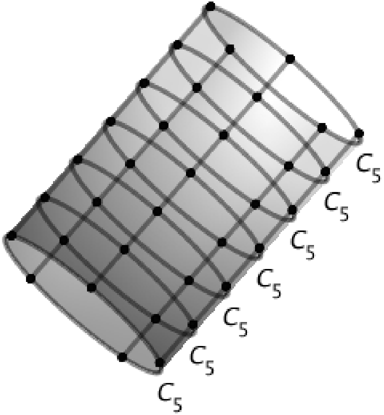
The number of near-perfect matchings in the graph depends on the vacancy location. If the vacancy were located on the boundary of the $C_3 \times P_3$ graph, then the number of near-perfect matchings would be equal to 5.



(We consider labeled graphs, so 2 matchings on the slide are different.)

The structure of Cartesian products

Consider the graph $G_{m,n} = C_m \times P_n$. This graph can be viewed as n copies of C_m placed sequentially with edges joining the corresponding vertices of C_m.



We number all the cycles C_m by integers from 1 to n. The ordinal number put in parentheses will be indicated as superscript. The union of graphs $\cup_{k=1}^{n} C_m^{(k)}$ forms a spanning subgraph of $G_{m,n}$.

Profile definition

Let $\mathcal{V}$ to be the set of vertices of $G_{m,n}$ and $K_m^v(n),\ (v \in \mathcal{V})$ – the number of near-perfect matchings on the cylinder when vacancy is fixed in the node v.

We denote by $\mathcal{V}_k$ the set of vertices of the cycle $C_m^{(k)}$. Consider two graphs $G_{m,n} - v_k'$ and $G_{m,n} - v_k''$, where $v_k', v_k'' \in \mathcal{V}_k,\ v_k' \neq v_k''$. Since these graphs are isomorphic, then $K_m^{v_k'}(n) = K_m^{v_k''}(n)$.

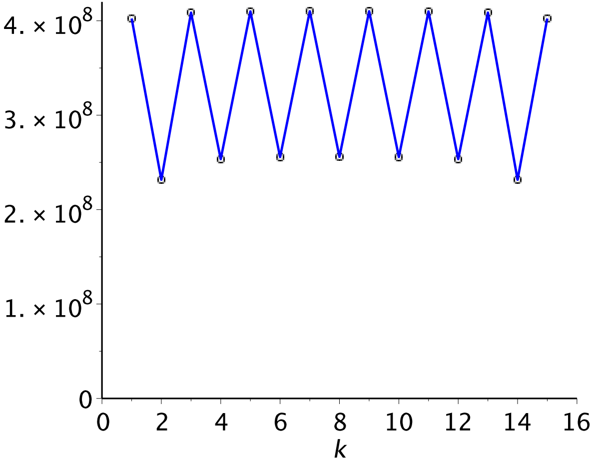
Let's introduce the notation $\hat{K}_m^{(k)}(n) = K_m^{v_k}(n)$, $(v_k \in \mathcal{V}_k)$. The set of values of $\hat{K}_m^{(k)}(n)$, $(k = 1, 2, \ldots, n)$ will be called the profile of near-perfect matchings on cylinder $G_{m,n}$.

The study is focused on key properties of $\hat{K}_m^{(k)}(n)$. This function satisfies the simple symmetry relation

$$\hat{K}_m^{(k)}(n) = \hat{K}_m^{(n+1-k)}(n),\ (k = 1, 2, \ldots, \lfloor n/2 \rfloor).$$

Profile on the cylinder $C_5 \times P_{15}$

Values of $\hat{K}_5^{(k)}(15)$.



Applications in statistical physics

A well-known monomer-dimer problem.

- Y. Kong. *Monomer-dimer model in two-dimensional rectangular lattices with fixed dimer density.* Physical Review E, 2006, vol.74, art.061102(15).

  Extensive data sets for the number of matchings on cylinders for $m \leqslant 17$.

- F.Y. Wu, W.-J. Tzeng and N.Sh. Izmailian. *Exact solution of a monomer-dimer problem: A single boundary monomer on a nonbipartite lattice.* Physical Review E, 2011, vol.83, art.011106(6).

  Closed form expression for $\hat{K}_m^{(1)}(n)$.

Main topics of research

- Significantly extend the set of known values of $\hat{K}_m^{(k)}(n)$ in a wide range of cylinder parameters (profiles for $m \leqslant 19$).
- To explore close connections between perfect and near-perfect matchings on cylinders of the same parameter m (establish recurrence relations for fixed values of m).
- Give exact or approximate values of the coefficients of asymptotic expansions for the total number of near-perfect matchings for different values of m.

A mechanical counting method by Wilf

H.S. Wilf. *A mechanical counting method and combinatorial applications.* Journal of Combinatorial Theory, 1968, vol.4, pp.246-258.

Let G be an undirected graph, without loops or mutliple edges, on n vertices. Let A be the $n \times n$ vertex adjacency matrix of G, $A = (a_{i,j})$. If n is even, consider the homogeneous polynomial

$$\left(\sum_{i,j=1}^{n} a_{i,j} x_i x_j \right)^{n/2}$$

The coefficient of $x_1 x_2 \cdots x_n$ counts each 1-factor exactly $2^{n/2}(n/2)!$ times.

Demo version of Maple code (one thread)

[Some kinds of graphs can be produced by direct calls of library functions.

```
>with(GraphTheory):
```

[Both cylinder parameters must be odd.

```
>m:=11:Cn:=135:
```

[Generate the graph and keep the list of its vertices.

```
>G:=CartesianProduct(PathGraph(Cn),CycleGraph(m)):
 V:=Vertices(G):
```

[Implementation of the algorithm by Wilf for a fixed vacancy location.

```
>Vacancy:=proc(v)
   local Gr,VV,n,x,g,h,i,vv;global G,Match;
   Gr:=DeleteVertex(G,v):
   VV:=Vertices(Gr):
   n:=NumberOfVertices(Gr):
```

Demo version of Maple code (continuation)

```
    g:=h(x[VV[1]])^(n/2):
    for i to n-1 do
      g:=eval(diff(g,x[VV[i]]),[x[VV[i]]=0,
         h(x[VV[i]])=h(x[VV[i+1]]),
         'diff'(h(x[VV[i]]),x[VV[i]])=add(x[vv],
         vv=Neighbors(Gr,VV[i]))]);
      x[VV[i]]:=0
    od:
    return diff(g,x[VV[n]])/(n/2)!
  end:
```

[Profile evaluation

```
>Match:=[seq(Vacancy(V[ind]),ind=1..nops(V),m)]:
```

[Do not forget to save the results on your disk!

```
>save Match, cat("E:\\C",m,"xP",Cn,".par");
```

Description superscripts (m is odd)

B – vacancy occurs on the boundary
$\hat{K}_m^B(n) = \hat{K}_m^{(1)}(n)$ – the number of near-perfect matchings on $C_m \times P_{2n+1}$ graph with one fixed vacant node on the boundary.
$G_m^B(z) = \sum_{n=0}^{\infty} \hat{K}_m^B(n) z^n$ – generating function for the sequence $\{\hat{K}_m^B(n)\}$.

P – perfect matchings
$K_m^P(n)$ – the number of perfect matchings on $C_m \times P_{2n}$ graph.
$G_m^P(z) = \sum_{n=0}^{\infty} K_m^P(n) z^n$ – generating function for the sequence $\{K_m^P(n)\}$.

N – near-perfect matchings
$\hat{K}_m^N(n) = m \cdot \sum_{k=1}^{n} \hat{K}_m^{(k)}(n)$ – the total number of near-perfect matchings on $C_m \times P_{2n+1}$ graph.
$G_m^N(z) = \sum_{n=0}^{\infty} \hat{K}_m^N(n) z^n$ – generating function for the sequence $\{\hat{K}_m^N(n)\}$.

Computational environment

- CPU: Core i7 980, 3.3GHz (6 cores)
- RAM: 24GiB (cache L3 12MiB)
- OS: Windows 7 professional 64-bit
- CAS: Maple 17.02 64-bit

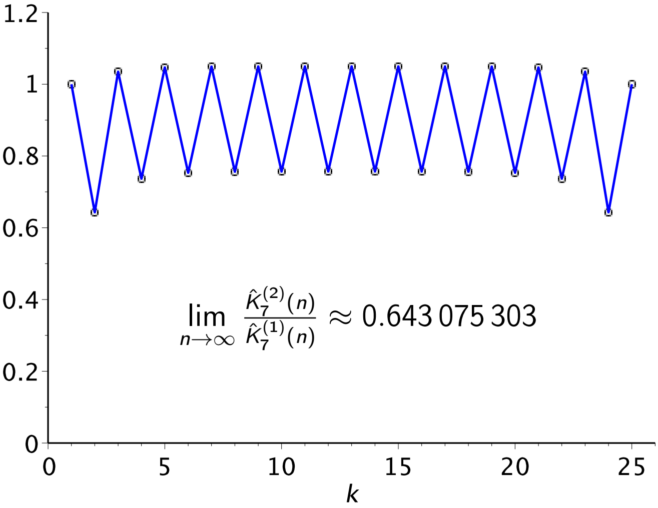
Available facilities make it possible to find linear recurrence relations and generating functions $G_m^B(z)$, $G_m^N(z)$ on cylinders for $m \leqslant 13$. Profiles of near-perfect matchings were evaluated on graphs $C_m \times P_n$ for $15 \leqslant m \leqslant 19$ and $n \leqslant n^*$, where

$$n^* = \begin{cases} 135, & \text{if } m = 15, 17, \\ 65, & \text{if } m = 19. \end{cases}$$

The principal term of asymptotic expansion of $\hat{K}_m^N(n)$ $(n \to \infty)$ was obtained for all the studied values of m.

Normalized profile on cylinder $C_7 \times P_{25}$

$$\max_{1\leqslant k\leqslant 25} \frac{\hat{K}_7^{(k)}(25)}{\hat{K}_7^{(1)}(25)} = \frac{\hat{K}_7^{(13)}(25)}{\hat{K}_7^{(1)}(25)} \approx 1.050, \quad \min_{1\leqslant k\leqslant 25} \frac{\hat{K}_7^{(k)}(25)}{\hat{K}_7^{(1)}(25)} = \frac{\hat{K}_7^{(2)}(25)}{\hat{K}_7^{(1)}(25)} \approx 0.643$$



$$\lim_{n\to\infty} \frac{\hat{K}_7^{(2)}(n)}{\hat{K}_7^{(1)}(n)} \approx 0.643\,075\,303$$

Cylinders $C_3 \times P_{2n+1}$

$$G_3^B(z) = \frac{1}{1-5\,z+z^2},\, G_3^P(z) = \frac{1-z}{1-5\,z+z^2}$$

$$\hat{K}_3^B(n) = \frac{1}{3}\left(K_3^P(n+1) - K_3^P(n)\right)$$

$$\lim_{n\to\infty} \hat{K}_3^B(n)/K_3^P(n) = \frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}}$$

$$G_3^N(z) = 3\frac{1+2\,z-z^2}{\left(1-5\,z+z^2\right)^2}$$

$$\hat{K}_3^N(n) = \frac{27n+17}{21}K_3^P(n+1) - \frac{15n+5}{21}K_3^P(n)$$

$$\begin{aligned}\hat{K}_3^N(n) = \quad & \tfrac{1}{7}\left(\left(2+\sqrt{21}\,\right)n+2+\tfrac{11}{\sqrt{21}}\right)\left(\tfrac{\sqrt{7}+\sqrt{3}}{2}\right)^{2n+2} + \\ & \tfrac{1}{7}\left(\left(2-\sqrt{21}\,\right)n+2-\tfrac{11}{\sqrt{21}}\right)\left(\tfrac{\sqrt{7}-\sqrt{3}}{2}\right)^{2n+2}\end{aligned}$$

Cylinders $C_5 \times P_{2n+1}$

$$G_5^B(z) = \frac{1-z^2}{1-19z+41z^2-19z^3+z^4}$$

$$G_5^P(z) = \frac{(1-z)(1-7z+z^2)}{1-19z+41z^2-19z^3+z^4}$$

$$\hat{K}_5^B(n) = \frac{1}{45}\left(K_5^P(n+2)+K_5^P(n-1)\right) - \frac{11}{45}\left(K_5^P(n+1)+K_5^P(n)\right)$$

$$\lim_{n\to\infty} \hat{K}_5^B(n)/K_5^P(n) = \frac{\sqrt{5}+\sqrt{30+2\sqrt{205}}+\sqrt{41}}{4\sqrt{5}}$$

$$G_5^N(z) = 5\frac{1+10z-56z^2+84z^3-24z^4-10z^5+z^6}{(1-19z+41z^2-19z^3+z^4)^2}$$

$$\hat{K}_5^N(n) = \tfrac{1}{615}\left(584n+\tfrac{1232}{3}\right)K_5^P(n+1) - \tfrac{1}{123}\left(80n+\tfrac{6752}{15}\right)K_5^P(n) - \tfrac{1}{123}\left(\tfrac{352n}{5}-\tfrac{1288}{3}\right)K_5^P(n-1) + \tfrac{1}{123}\left(\tfrac{23n}{5}-\tfrac{73}{3}\right)K_5^P(n-2)$$

Closed form expressions for odd m

$$c_j(m) = \sin\left(\frac{\pi(2j-1)}{m}\right) + \sqrt{1+\sin^2\left(\frac{\pi(2j-1)}{m}\right)},$$

$$c_j(m)\bar{c}_j(m) = -1,\ D_m^2 = \frac{1}{2}\left((\sqrt{2}+1)^m - (\sqrt{2}-1)^m\right).$$

Number of perfect matchings on cylinder $C_m \times P_{2n}$

$$K_m^P(n) = \frac{1}{D_m}\prod_{j=1}^{\lfloor m/2 \rfloor}\left(c_j(m)^{2n+1} - \bar{c}_j(m)^{2n+1}\right).$$

Number of near-perfect matchings on cylinder $C_m \times P_{2n+1}$ with vacancy on the boundary

$$\hat{K}_m^B(n) = \frac{1}{D_m\sqrt{m}}\prod_{j=1}^{\lfloor m/2 \rfloor}\left(c_j(m)^{2n+2} - \bar{c}_j(m)^{2n+2}\right).$$

$$\lim_{n\to\infty}\hat{K}_m^B(n)/K_m^P(n) = \frac{\Lambda(m)}{\sqrt{m}},\ \text{where } \Lambda(m) = \prod_{j=1}^{\lfloor m/2 \rfloor} c_j(m).$$

A few conjectures and open problems

Conjecture 1. For *all* odd values of m sequences $\{\hat{K}_m^{(k)}(n)\}$ and $\{K_m^P(n)\}$ obey the same recurrence relation.

Conjecture 2. For *all* odd values of m denominator $G_m^N(z)$ is always the square of denominator $G_m^P(z)$.

Open problem 1. The results of computational experiments indicate that

$$\lim_{m\to\infty}\lim_{n\to\infty} K_m^{(2)}(n)/K_m^{(1)}(n) \approx 0.727.$$

Give closed form expression for this value.

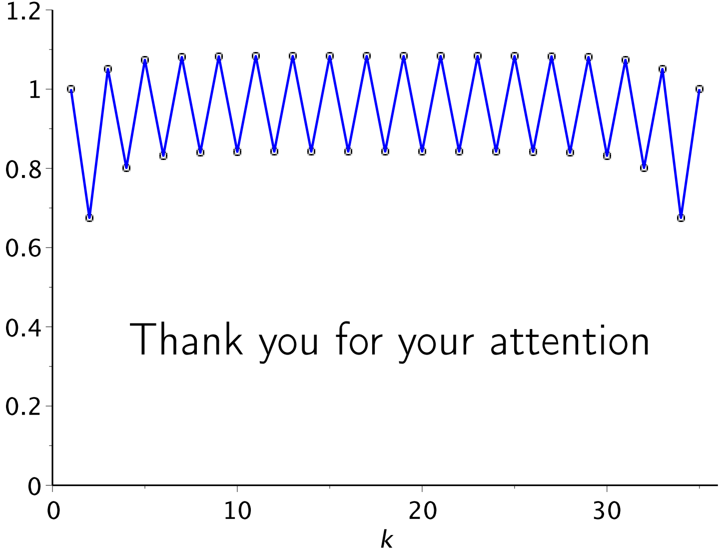
Open problem 2. Find the asymptotics of the quantity

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant n}\frac{\hat{K}_m^{(k)}(n)}{\hat{K}_m^{(1)}(n)}$$

for large m.

Conclusion

Normalized profile on cylinder $C_9 \times P_{35}$



Thank you for your attention