Non-perturbative gauge-fixing of compact gauge fields on the lattice

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The talk is divided into two parts :

- Part I is about the scheme of gauge-fixing for Abelian theory.
- Part II sets down the groundwork for the eBRST scheme for non-Abelian theories. The work is still in progress.

Motivations for gauge-fixing on the lattice

Wilson $\longrightarrow$ **Manifestly gauge-invariant** formalism on a discrete spacetime Euclidean lattice for gauge theories.

Group-valued link fields $\Rightarrow$ algebra-valued gauge fields are compact.

Partition function $\longrightarrow$ **gauge-invariant well-defined integral** with a **gauge-invariant Haar measure**.

Hence, gauge-fixing generally not required on the lattice.

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Nielsen-Ninomiya theorem : **fermion species doubling**
$\Rightarrow$ Need to explicitly break chiral symmetry by lattice fermions

Lattice chiral gauge theories break gauge invariance.

Two ways to construct a genuinely ChLGT :

- modify chiral symmetry on lattice $\longrightarrow$ **Ginsparg-Wilson relation** [Luescher 1999,2000]
- try to mend explicitly broken theory
  - longitudinal gauge degrees of freedom (*lgdof*) couple with the physical degrees of freedom due to the broken symmetry $\Rightarrow$ destroy chiral nature of fermion spectrum
    [Bock, De, Smit, Nucl. Phys. B388, 243 (1992), Golterman, Petcher, Smit, Nucl. Phys. B370, 51 (1992)]
  - non-perturbative **rough gauge problem**

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Way to tackle $\rightarrow$ give dynamics to the *lgdof* $\rightarrow$ gauge-fixing.

Detour

Wilson's 1973 paper showed confinement for non-abelian theories **on the lattice** $\rightarrow$ similar calculation holds for U(1).

Simulations showed a **weak gauge-coupling Coulomb phase** (with free massless photons) and a **strong coupling phase** with nontrivial properties (gaugeballs, etc.) separated by a **weak first-order transition** at $g \approx 1$.

$\Rightarrow$ **Continuum limit cannot be taken.**

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$\Rightarrow$ **Continuum limit cannot be taken.**

Extensions of the pure $U(1)$ theory on the lattice have showed continuous phase transitions where continuum limit can be taken. All these continuum limits are the expected trivial theory of free photons $\rightarrow$ no non-triviality observed.
[Vettorazzo, de Forcrand, Nucl. Phys. B 686 (2004)]

Fixing the gauge

We need to tame the redundant degrees of freedom.

Standard **Fadeev-Popov** $\longrightarrow$ **fails** non-perturbatively.
Neuberger's theorem $\Rightarrow$ BRST symmetry renders any gauge-invariant observable an indeterminate $0/0$ form (due to cancellation among lattice Gribov copies).

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Goals:

- break BRST symmetry on the lattice.
- to achieve **renormalizable Lorentz gauge** in the continuum,

$$S_{GF} = \frac{1}{2\xi}(\partial_\mu A_\mu)^2$$

enabling weak-coupling perturbation theory (**WCPT**) around a **unique minimum**.

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Naive transcription on the lattice fails

- No unique vacuum ; standard perturbation theory fails
- Dense set of Gribov copies
- Neuberger's theorem : Exact BRST symmetry leading to 0/0 form.

- The proposed action for the compact gauge-fixed U(1) theory, where the ghosts are free and decoupled:

$$S[U] = S_g[U] + S_{gf}[U] + S_{ct}[U].$$

where $S_g[U] = \frac{1}{g^2} \sum_{x\, \mu<\nu} (1 - \mathrm{Re}\, U_{\mu\nu x})$ is usual Wilson plaquette action.

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- Golterman and Shamir proposed the higher derivative (HD) gauge-fixing term

$$S_{gf}(U) \equiv S_{HD}(\phi, U)|_{\phi_x = I} \quad ; \quad S_{HD} = \tilde{\kappa} \left(\sum_{x,y} \phi_y^\dagger \Box_{yx}^\dagger \Box_{xy} \phi_y - \sum_x B_x B_x \right)$$

$$\Box_{xy}(U) = \sum_\mu \left(\delta_{y,x+\mu} U_{x\mu} + \delta_{y,x-\mu} U_{x-\mu,\mu}^\dagger - 2\delta_{yx} \right)$$

$$B_x = \sum_\mu \left(\frac{V_{x\mu} + V_{x-\mu,\mu}}{2} \right)^2, V_{x\mu} = \frac{1}{2i} \left(\phi_x^\dagger U_{x\mu} \phi_{x+\mu} - \text{h.c.} \right)$$

- Equivalent "Higgs" (with ϕ fields) and vector picture(without) $\rightarrow$ related by a gauge transformation. We work in the vector picture.

The Regularization : Features

The HD term satisfies all desirable properties discussed earlier.

$S_{gf} \to \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 +$ irrelevant terms, where $\boldsymbol{\alpha = 1/2\tilde{\kappa} g^2}$

The action has an unique absolute minimum at $U_{x\mu} = \exp igA_{\mu x} = I$, thus validating weak coupling perturbation around $g = 0$ and $\tilde{\kappa} = \infty$.

Determine the form of the counterterms needed to recover the gauge symmetry by power counting.
It turns out that the most important gauge counterterm is the gauge field mass counterterm, given by,

$$S_{ct} = -\kappa \sum_{\mu x} \left(U_{\mu x} + U^\dagger_{\mu x} \right) .$$

It alone leads to a continuous phase transition where the ϕ fields decouple and the gauge symmetry is recovered.

Previous work

Constant field approximation of the classical potential shows a **phase transition** at $\kappa = 0$ between a **broken gauge symmetry phase (FM)** and a phase with **broken Euclidean symmetry (FMD)**, at which point the gauge symmetry is restored.

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The HD gauge-fixing proposal, studied extensively in the weak coupling region (**Bock et al 2000**) verifies the above result. For sufficiently large coefficient $\tilde{\kappa}$, there is a novel continuous phase transition between FM and FMD phases. Approaching the transition from the FM-side, we obtain free massless photons only and the scalar fields ($lgdof$) decouple. These results have been explicitly verified in weak gauge coupling region using both perturbative analysis and by numerical simulations.

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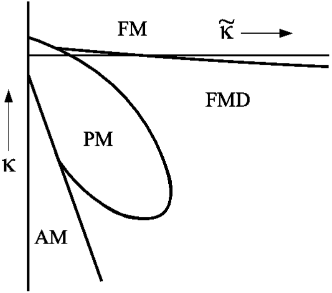
Manifestly local abelian chiral gauge theories on the lattice was succesfully shown for Wilson fermions (**Bock et al 1998**) and also for lattice domain wall fermions (**Basak, De 2001**) with the HD gauge-fixing proposal.

Previous work : Weak coupling phase diagram



Schematic phase diagram for weak coupling at a fixed $g < 1$.

Earlier results have been done with the above action either in the reduced limit or in the weak coupling region in very small lattice sizes.

The question now arises as to what happens in the strong coupling region. Knowledge about a broad range of gauge coupling is also required to understand the equivariant BRST gauge-fixing proposal for non-Abelian gauge theories on the lattice.

Our strong coupling study : Algorithms used

- Hybrid Monte Carlo algorithm : Needed due to the HD term in the action. Simulation done in lattice volumes $10^4, 12^4, 16^4, 20^4, 24^4$

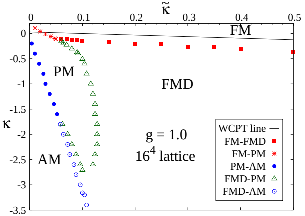
  Multihit Metropolis algorithm was tried but it fails as the gauge-fixing coupling gets stronger i.e. due to the strong influence of HD term.

- **Observables**:

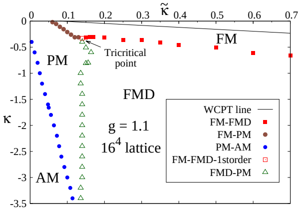
$$\mathbf{E}_\kappa = \frac{1}{4L^4}\left\langle \sum_{x,\mu} \mathrm{Re}\, U_{\mu x} \right\rangle, \quad \mathbf{V} = \left\langle \sqrt{\frac{1}{4}\sum_{\mu}\left(\frac{1}{L^4}\sum_{x} \mathrm{Im}\, U_{\mu x}\right)^2} \right\rangle$$

- **Quenched chiral condensate** using staggered fermions
- **Gauge & scalar field propagator**

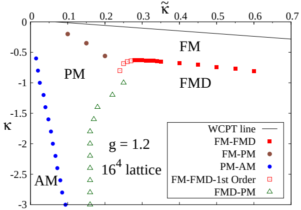
Results: Strong coupling phase diagram / Tricritical point



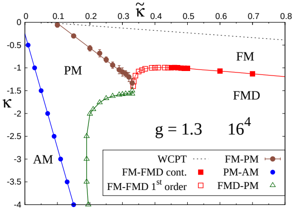
$g = 1.0$



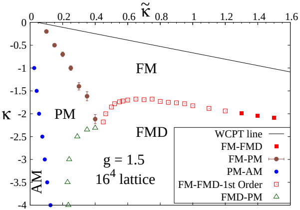
$g = 1.1$



$g = 1.2$



$g = 1.3$



$g = 1.5$

Existence of **tricritical point** where order of FM-FMD transition changes.

Continuous FM-FMD transition, for sufficiently large value of $\tilde{\kappa}$, with same properties as in the weak gauge coupling has been obtained $\rightarrow$ indicates that features governed are by the same perturbative fixed point.

First order vs Continuous phase transition

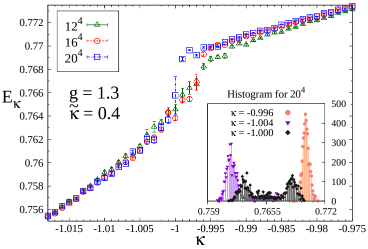


Figure: First order transition. Histogram shows double peak at $\kappa = -1.000$ which is a failsafe signature for first order transitions.

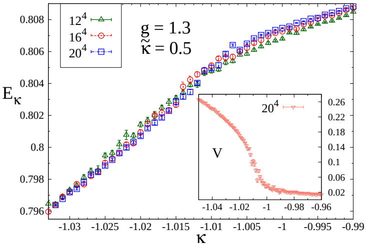
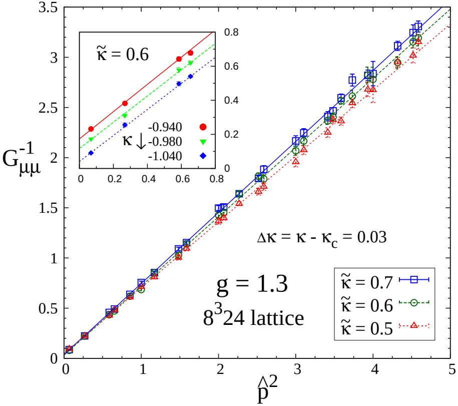


Figure: Continuous transition. Inset shows the transition for observable V

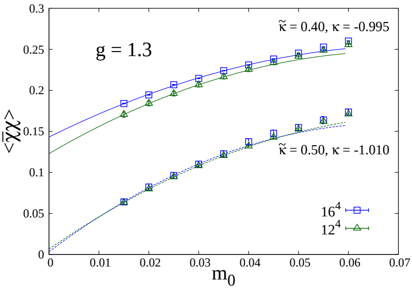
Inverse photon propagator



Inverse photon propagator near FM-FMD transition

- The FM-FMD transition is approached from the FM side.
- Slope of the straight line fits increases with increasing $\tilde{\kappa}$. Non-unity slope suggests field renormalization constant Z but trend suggests **approach towards the perturbative point with free photons**.
- The vanishing y-intercept shows **zero photon mass** and thus **recovery of gauge symmetry** near the FM-FMD transition.

Results: Quenched chiral condensate



- Near the tricitical point, quenched chiral condensate was calculated with KS fermions.
- Chiral transition occurs near the tricritical point at our precision.
- Nontrivial physics if nonzero chiral condensate occurs in the continuous side of the FM-FMD transition → confirmation needed on bigger volumes. Ruled out at present.

Conclusions

- The broad scan of the phase diagram reveals that the physics at strong coupling for sufficiently large $\tilde{\kappa}$ is the same as the weak coupling region.
- Existence of a tricritical point which may lead to nontrivial physics in the theory. Probing the physics near the tricritical point is difficult and requires much more work.
- The chiral phase transition occurs at the tricritical point at our precision thus ruling out possible non-triviality.
- This scheme of abelian gauge-fixing is crucial for the overall success of the gauge-fixing approach to chiral gauge theories, the equivariant BRST scheme of gauge-fixing, proposed by Schaden, Golterman and Shamir.
- Key results for one value of strong coupling has been published. **(Phys. Rev. D 93, 114504 (2016))**

Gauge-fixing Yang-Mills theory

Covariantly gauge-fixed YM theory

- transverse gauge coupling $\boldsymbol{g}$
- longitudinal gauge coupling $\tilde{\boldsymbol{g}}$ ($\tilde{g} = \xi g^2$)

$\Rightarrow$ both asymptotically free

[M. Golterman and Y. Shamir, Phys. Rev. D 73 (2006)]

BRST symmetry ensures physics independent of $\tilde{\boldsymbol{g}}$

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Question : Are the two sectors independent even non-perturbatively?

First, construct non-perturbatively gauge-fixed theory
usual choice is on the lattice
but standard BRST formalism fails $\longrightarrow$ Neuberger's theorem

Proposal: Schaden first proposed the **eBRST gauge-fixing scheme** on the lattice for $SU(2)$ gauge theory in 1998. **Golterman** and **Shamir** later extended the idea for general $SU(N)$ theories with extended eBRST in the context of lattice chiral gauge theories in 2004.

[M. Schaden, Phy. Rev. D59, 014508, M. Golterman, Y. Shamir, Phy. Rev. D70, 094506]

eBRST in the continuum

The Yang-Mills Lagrangian with gauge coupling g is

$$\mathcal{L}_{YM} = \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}^2), \quad iF_{\mu\nu} = [D_\mu(V), D_\nu(V)],$$
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Gauge-fixing is done in the coset space G/H leaving atleast the maximal Abelian subgroup $H \subset G$ invariance of the action.

$$V_\mu = V_\mu^a T^a = A_\mu^i T^i + W_\mu^\alpha T^\alpha$$

where index a runs over the generators of G,
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The gauge-fixing condition is chosen as

$$\mathcal{F}(V) = \mathcal{D}_\mu(A)W_\mu \equiv \partial_\mu W_\mu + i[A_\mu, W_\mu]$$

where $\mathcal{D}_\mu(A)$ is a covariant derivative w.r.t. to H in the adjoint representation. The ghost fields and the auxiliary field now reside in the coset space $\mathcal{G}/\mathcal{H}$ (curly for algebra):

$$C = C^\alpha T^\alpha, \quad \overline{C} = \overline{C}^\alpha T^\alpha \quad \text{and} \quad b = b^\alpha T^\alpha$$

BRST vs eBRST

BRST

- $\delta_B \Psi = -iC\Psi, \quad \delta_B \Psi^\dagger = iC^a \Psi^\dagger T^a,$
- $\delta_B V_\mu = \mathcal{D}_\mu(V)C,$
- $\delta_B C = -iC^2,$
  $\delta_B \overline{C} = -ib,$
- $\delta_B b = 0,$
- Condition of nilpotency : $\delta_B^2 f = 0$

eBRST

- $s\Psi = -iC\Psi, \quad s\Psi^\dagger = iC^\alpha \Psi^\dagger T^\alpha,$
- $sA_\mu = i[W_\mu, C]_{\mathcal{H}},$
  $sW_\mu = \mathcal{D}_\mu(A)C + i[W_\mu, C]_{\mathcal{G}/\mathcal{H}}$
- $sC = (-iC^2)_{\mathcal{G}/\mathcal{H}} = -iC^2 + X$, where
  $X \equiv (-iC^2)_{\mathcal{H}},$
  $s\overline{C} = -ib,$
- $sb = [X, \overline{C}],$
- Equivariant nilpotency : $s^2 = \delta_X$, a gauge transformation in H with parameter $X \in \mathcal{H}$

General eBRST gauge-fixing term

eBRST and H transformations commute with each other.

Generic form of gauge-fixing action $S_{gf} = s\Sigma$ where Σ is H-invariant.
Action is both invariant under eBRST and H transformations but breaks BRST symmetry.

The gauge-fixing Lagrangian is given as

$$\mathcal{L}_{gf} = s\,\mathrm{tr}(2\overline{C}\mathcal{F} + i\xi g^2\overline{C}b)$$

Upon simplification, we obtain a **4-ghost term** and a **bilinear ghost term**. The 4-ghost term saves the partition function from vanishing as in the case of BRST.

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But this is not the most general action possible. An extended eBRST theory can be constructed with eBRST and anti-eBRST symmetry. Reduces to above form for our case $SU(2)/U(1)$ (Special Class).

- $SU(2)/U(1)$ gauge-fixed Lagrangian in the continuum after integrating out the auxiliary field b

$$\mathcal{L}_{gf} = \frac{1}{\xi g^2}\mathrm{tr}(\mathcal{D}_\mu(A)W_\mu)^2 + \mathcal{L}_{gh}^{(2)} + \xi g^2 \mathcal{L}_{gh}^{(4)}$$

$$\mathcal{L}_{gh}^{(2)} = -2\mathrm{tr}(\overline{C}\mathcal{D}_\mu(A)\mathcal{D}_\mu(A)C) + 2\mathrm{tr}([W_\mu, \overline{C}][W_\mu, C])$$

$$\mathcal{L}_{gh}^{(4)} = -\mathrm{tr}(\widetilde{X}^2), \widetilde{X} = \mathrm{i}\{C, \bar{C}\}$$

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- **Action on the lattice**

$$\mathcal{S}_{gf} = \frac{1}{\xi g^2}\mathrm{tr}\sum_x (\mathcal{D}_\mu^- \mathcal{W}_{x\mu})^2 - \xi g^2 \mathrm{tr}\sum_x (\widetilde{X}^2)$$
$$- 2\mathrm{tr}\sum_x \left(\left[U_{x\mu}T_3U_{x\mu}^\dagger, \mathcal{D}_\mu^+\overline{C}_x\right]\left[T_3, \mathcal{D}_\mu^+ C_x\right]\right) + i\mathcal{W}_{x\mu}\left\{\overline{C}_x, \mathcal{D}_\mu^+ C_x\right\}$$

where $\mathcal{W}_{x\mu} = -[U_{x\mu}T_3U_{x\mu}^\dagger, T_3] = W_{x\mu} + O(V^2), T_a = \sigma_a/2$, lattice covariant derivatives $\mathcal{D}_\mu^+\Phi_x = U_{x\mu}\Phi_{x+\mu}U_{x\mu}^\dagger - \Phi_x, \mathcal{D}_\mu^-\Phi_x = \Phi_x - U_{x-\mu,\mu}^\dagger \Phi_{x-\mu}U_{x-\mu,\mu}$.

Important relation used $\sum_i \mathrm{tr}([T_i, A][T_i, B]) = -\mathrm{tr}(A_{\mathcal{G}/\mathcal{H}}B_{\mathcal{G}/\mathcal{H}})$

Action on the lattice; Ghost matrix

- The 4-ghost term is tackled by introducing an **auxiliary field** $\rho \in \mathcal{H}$. The action becomes

$$\mathcal{S}_{gf} = \frac{1}{2\xi g^2} \sum_{x\alpha} (\mathcal{D}_\mu^- \mathcal{W}_{x\mu})_\alpha^2 + \frac{1}{2\xi g^2} \sum_x \rho_x^2 + \sum_{xy\alpha\beta} \overline{C}_{x\alpha} M_{x\alpha,y\beta} C_{y\beta}$$

- The ghost matrix $M_{x\alpha,y\beta} = \Omega_{x\alpha,y\beta}(U) + R_{x\alpha,y\beta}(\rho)$ is real, with Ω being symmetric and $R_{x\alpha,y\beta} = \delta_{xy}\rho_x f_{3\alpha\beta}$ being antisymmetric in the indices α and β.

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- The ghost matrix is implemented in the HMC algorithm in the following way. Integrating out ghost fields, we get

$$\int \mathcal{D}C\mathcal{D}\overline{C} \exp(-\overline{C}MC) = \det M = |\det M| \, \mathrm{sign}(\det M)$$

Since entries of M are real, $|\det M|$ can be simulated using HMC by introducing a real "pseudo-ghost" field ϕ,

$$|\det M| = \sqrt{\det(MM^T)} = \int \mathcal{D}\phi \exp\left(-(1/2)\phi^T (MM^T)^{-1} \phi\right).$$

- The sign of the determinant is an important part of the whole scheme.

Conclusion

- The eBRST scheme of gauge-fixing is a very novel approach to address the problem of non-abelian lattice chiral gauge theories. Present work is only with pure gauge. We intend to study the phase diagram which emerges from such theories.
- Coding is a challenging task since keeping track of the sign of the determinant will be a very difficult thing as it essentially boils down to tracking the zero crossing of the smallest eigenvalues. We intend to use some kind of deflation techniques with HMC.
- Ultimately, the abelian part of the theory has to be gauge-fixed by the HD action described in the previous section.

Thank you for your kind attention

Backup slides

Constant Field Approximation

The classical potential, obtained as leading order term in the perturbative expansion of $U_{\mu x} = \exp igA_{\mu x}$ with constant field approximation(no derivative terms) around $U_{\mu x} = 1$, is

$$V_{cl} = \kappa\left[g^2 \sum_\mu A_\mu^2\right] + \frac{g^4}{2\alpha}\left[\left(\sum_\mu A_\mu^2\right)\left(\sum_\mu A_\mu^4\right)\right]$$

For $\kappa > 0$, the gauge boson is massive and V_{cl} has a minimum at $A_\mu = 0$. A broken phase called FM phase.
For $\kappa < 0$, the minimum of V_{cl} shifts to a nonzero value:

$$A_\mu = \pm\left(\frac{\alpha|\kappa|}{3g^2}\right)^{\frac{1}{4}} \qquad \text{for all } \mu$$

implying an unusual phase with broken rotational symmetry in addition to the broken gauge symmetry – directional ferromagnetic phase (FMD).
For $\kappa = 0 \equiv \kappa_c$, the gauge boson becomes massless with the minimum of V_{cl} still being the same $\Rightarrow$ phase transition at this point.

The action is also invariant under a ghost flip symmetry defined as $FC = \overline{C}, F\overline{C} = -C$ ($F\Phi = \Phi$ for all physical fields)

This introduces the concept of anti-eBRST variation $\bar{s}$ whose transformation rules are obtained by ghost flip of the eBRST rules. Baulieu & Thierry-Mieg introduced the concept of anti-BRST in 1982.

We thus have the extended eBRST algebra

$$s^2 = \delta_X, \quad \bar{s}^2 = \delta_{\overline{X}}, \quad \{s, \bar{s}\} = \delta_{\tilde{X}}$$

where $X = (iC^2)_{\mathcal{H}}$, $\quad \overline{X} = (i\overline{C}^2)_{\mathcal{H}}$, $\quad \tilde{X} = i\left\{C, \overline{C}\right\}_{\mathcal{H}}$

The most general action can be written as follows

$$\mathcal{L}_{gf} = -s\bar{s}\,\mathrm{tr}(W^2 + \xi g^2 \overline{C} C)$$

which is invariant under eBRST, anti-eBRST, ghost flip and H gauge symmetry.

For the case of $SU(2)/U(1)$ (example of Special Class), the coset structure constants are all equal to zero and a lot of simplifications occur.

An observable $\Theta(U)$ is calculated using a slightly modified partition function Z' in the following way :

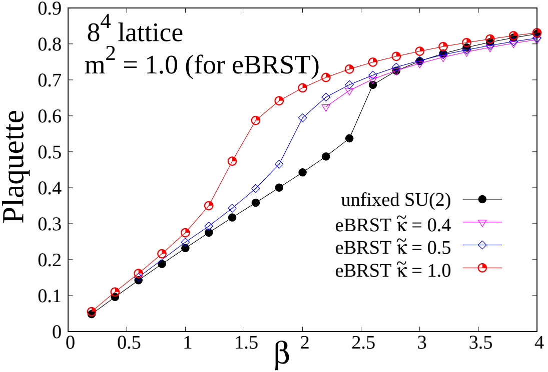
$$Z \equiv \int \mathcal{D}U\mathcal{D}\phi \exp\left(-[\mathcal{S}_W + \mathcal{S}'_{gf} + \frac{1}{2}\phi^T(M^TM)^{-1}\phi]\right) \mathrm{sign}(\det M)$$

$$Z' \equiv \int \mathcal{D}U\mathcal{D}\phi \exp\left(-[S_W + S_{GF} + \frac{1}{2}\phi^T(M^TM)^{-1}\phi]\right)$$

$$\begin{aligned}\langle\Theta\rangle_Z &= \frac{1}{Z}\int \mathcal{D}U\mathcal{D}\phi \exp\left(-[S_W + S_{GF} + \frac{1}{2}\phi^T(M^TM)^{-1}\phi]\right) (\mathrm{sign}(\det M))\Theta(U) \\ &= \frac{Z'}{Z}\frac{1}{Z'}\int \mathcal{D}U\mathcal{D}\phi \exp\left(-[S_W + S_{GF} + \frac{1}{2}\phi^T(M^TM)^{-1}\phi]\right) (\mathrm{sign}(\det M))\Theta(U) \\ &= \frac{Z'}{Z}\langle(\mathrm{sign}(\det M))\Theta\rangle_{Z'}\end{aligned}$$

Now for $\Theta = 1$, we have $\langle(\mathrm{sign}(\det M))\rangle_{Z'} = Z/Z'$

$$\therefore \langle\Theta\rangle_Z = \frac{\langle(\mathrm{sign}(\det M))\Theta\rangle_{Z'}}{\langle(\mathrm{sign}(\det M))\rangle_{Z'}}$$



Plaquette vs β with Z'

From the *invariance* theorem, we expect, for gauge-invariant operator $\mathcal{O}$, $\langle \mathcal{O}(U) \rangle_{\text{unfixed}} = \langle \mathcal{O}(U) \rangle_{\text{eBRST}}$.

Without taking into account the sign changes, the figure demonstrates which regions of the phase space are affected to what degrees due to the sign change.

The matrix inversion becomes difficult with smaller $\tilde{\kappa}$ for values of β around the crossover region, which accounts for the missing data points.

The plot indicates that the eBRST gauge-fixed curves approach the unfixed curve as $\tilde{\kappa} \to 0$.