Ricci-flat spacetime admitting higher rank Killing tensors and the Eisenhart lift

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Outline

1. Killing tensor
2. The Eisenhart lift
3. Ricci–flat spacetimes from systems on pseudo-Euclidian plane
4. Self–dual metrics with maximally superintegrable geodesic flows

Objective:

- To construct a class of self–dual and geodesically complete spacetimes with maximally superintegrable geodesic flows which admit a second rank Killing tensor

Killing tensors

In Riemannian geometry, a Killing vector field $\xi^n(x)$ generates an infinitesimal transformation which leaves the form of the metric invariant

$$x'^n = x^n + \xi^n(x), \quad g'_{nm}(x) - g_{nm}(x) = -(\nabla_n \xi_m(x) + \nabla_m \xi_n(x)) = 0 \tag{1}$$

To each Killing vector there corresponds a first integral of the geodesic equations

$$\xi^n(x) g_{nm}(x) \frac{dx^m}{d\tau} \tag{2}$$

A Killing tensor of rank n is a totally symmetric tensor field $K_{i_1 \dots i_n}(x)$ which obeys a similar equation

$$\nabla_{(i_1} K_{i_2 \dots i_{n+1})}(x) = 0 \tag{3}$$

As there is no coordinate transformation associated with a Killing tensor which would leave the metric invariant, these somewhat enigmatic objects are attributed to hidden symmetries of spacetime. To each Killing tensor there corresponds a first integral of the geodesic equations which is of degree n in velocities

$$K_{i_1 \dots i_n}(x) \frac{dx^{i_1}}{d\tau} \dots \frac{dx^{i_n}}{d\tau} \tag{4}$$

Killing tensors

Applications of Killing tensors:

- They provide integrability of geodesic equations and help to solve them by quadrature
- They are essential for separating variables in the Hamilton–Jacobi and Schödinger equations, as well as in Klein–Gordon and Dirac equations formulated on curved backgrounds
- They help to identify a spacetime in accord with Petrov's classification
- They provide a means of studying integrable systems in curved space

Remark: no Ricci–flat spacetime admitting a Killing tensor of rank>2 was known until recently (Cariglia, Galajinsky, PLB'15).

Kerr metric in Boyer–Lindquist coordinates ($x^m = (t, r, \theta, \phi)$, $d\tau^2 = g_{nm} dx^n dx^m$)

$$d\tau^2 = dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{2Mr}{\rho^2} \left(dt - a \sin^2\theta d\phi\right)^2 - (r^2 + a^2) \sin^2\theta d\phi^2$$

$$\Delta = r^2 + a^2 - 2Mr, \qquad \rho^2 = r^2 + a^2 \cos^2\theta$$

The second rank Killing tensor (B. Carter, 1968; M. Walker, R. Penrose, 1970)

$$K_{mn} = Q_{mn} + r^2 g_{mn}, \qquad \nabla_{(k} K_{mn)} = 0,$$

$$Q_{mn} = \begin{pmatrix} -\Delta & 0 & 0 & a\Delta \sin^2\theta \\ 0 & \frac{\rho^4}{\Delta} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\Delta \sin^2\theta & 0 & 0 & -a^2 \Delta \sin^4\theta \end{pmatrix}.$$

For black hole solutions in $d > 4$ there appears a chain of rank–2 Killing tensors.

The Eisenhart lift

The Eisenhart lift (L. Eisenhart, 1929) is a geometric description of a dynamical system with n degrees of freedom such that the equations of motion following from the Lagrangian $\mathcal{L} = \frac{1}{2}\dot{x}_i\dot{x}_i - U(x)$ are embedded into the null geodesic equations in $(n+2)$–dimensional Lorentzian spacetime $\left(z^A = (t, s, x_1, \ldots, x_n)\right)$

$$d\tau^2 = g_{AB}(z)dz^A dz^B = -2U(x)dt^2 + 2dtds + \sum_{i=1}^{n}(dx_i)^2 \quad (5)$$

$$\frac{d^2z^A}{d\tau^2} + \Gamma^A_{BC}(z)\frac{dz^B}{d\tau}\frac{dz^C}{d\tau} = 0, \qquad g_{AB}(z)\frac{dz^A}{d\tau}\frac{dz^B}{d\tau} = 0 \quad (6)$$

Being rewritten in components, they read (c_1, c_2 are constants)

$$\frac{d^2x_i}{dt^2} + \partial_i U(x) = 0, \qquad \frac{dt}{d\tau} = c_1, \qquad \frac{ds}{dt} - 2U(x) = c_2 \quad (7)$$

$$\frac{1}{2}\sum_{i=1}^{n}\left(\frac{dx_i}{dt}\right)^2 + \frac{ds}{dt} - U(x) = 0 \quad (8)$$

The original dynamics is recovered by implementing a null reduction along s.

An integral of motion of the dynamical system which is a polynomial in velocities turns into a Killing tensor (multiplication by $\left(\frac{dt}{d\tau}\right)^n$)

$$I = \sum_{r=0}^{n} \alpha^{(r)}_{i_1 \dots i_r}(t,x) \frac{dx^{i_1}}{dt} \dots \frac{dx^{i_r}}{dt} \quad \rightarrow \quad K_{A_1 \dots A_n}(z) \frac{dz^{A_1}}{d\tau} \dots \frac{dz^{A_n}}{d\tau} \tag{9}$$

$$\nabla_{(A_1} K_{A_2 \dots A_{n+1})}(z) = 0 \tag{10}$$

Plenty of Lorentzian spacetimes admitting higher rank Killing tensors are known (G.W. Gibbons, T. Houri, D. Kubiznak, C. Warnick, C. Rugina, A.G, M. Cariglia, J.W. van Holten, P.A. Horváthy, P. Kosinski, P.M. Zhang). None of them is Ricci–flat.

The Eisenhart metric is Ricci–flat provided if the potential is a harmonic function

$$R_{tt} = \sum_{i=1}^{n} \partial_i \partial_i U(x) = 0 \tag{11}$$

None of the potentials compatible with integrals of motion cubic (or higher) in velocities known to date is described by a harmonic function.

Ricci–flat spacetimes from systems on pseudo-Euclidian plane

Remark: the constraint on the potential is changes if one alters the signature

Consider mechanics in $d = 2$ pseudo–Euclidean spacetime of signature $(1, 1)$

$$H = p_x p_y + U(x, y) \tag{12}$$

where (p_x, p_y) are momenta canonically conjugate to (x, y).

The Eisenhart metric of the ultrahyperbolic signature $(2, 2)$

$$d\tau^2 = g_{AB}(z) dz^A dz^B = -2U(x, y)dt^2 + 2dtds + 2dxdy \tag{13}$$

The Riemann tensor $(U_{xx} = \frac{\partial^2 U}{\partial x^2})$

$$R^s{}_{xtx} = U_{xx}, \qquad R^s{}_{xty} = U_{xy}, \qquad R^s{}_{ytx} = U_{xy}, \qquad R^s{}_{yty} = U_{yy} \tag{14}$$

$$R^x{}_{ttx} = -U_{xy}, \quad R^x{}_{tty} = -U_{yy}, \quad R^y{}_{ttx} = -U_{xx}, \quad R^y{}_{tty} = -U_{xy} \tag{15}$$

The Ricci tensor

$$R_{tt} = 2U_{xy} \tag{16}$$

Ricci–flat spacetimes from systems on pseudo-Euclidian plane

Thus if the metric is Ricci–flat provided the potential is an additive function

$$U(x,y) = u(x) + v(y) \tag{17}$$

(Anti)–self–dual spacetime is defined by

$$\frac{1}{2}\sqrt{g} g^{ml} g^{nr} \epsilon_{lrqs} R^{p}{}_{kmn} = \pm R^{p}{}_{kqs} \tag{18}$$

Self–dual metric

$$U_{xx} = 0, \qquad U_{xy} = 0 \quad \rightarrow \quad U(x,y) = a + bx + v(y) \tag{19}$$

Anti–self–dual metric

$$U_{yy} = 0, \qquad U_{xy} = 0, \quad \rightarrow \quad U(x,y) = a + by + u(x) \tag{20}$$

Mechanics in $d = 2$ pseudo–Euclidean spacetime of signature $(1,1)$ which admits a cubic integral of motion has been studied by Drach (Drach'35; Cariglia, Galaginsky'15).

Let us inquire whether a self–dual and geodesically complete spacetime can be constructed which admits a Killing tensor of valence two.

Special mechanics in $d = 2$ pseudo–Euclidean spacetime of signature $(1, 1)$

$$H = p_x p_y + v(y) + bx + a \tag{21}$$

Consider a polynomial quadratic in momenta

$$I_2 = A(x, y)p_x^2 + B(x, y)p_y^2 + C(x, y)p_x p_y + D(x, y) \tag{22}$$

Conservation of I_2 in time yields the system of linear partial differential equations

$$\begin{aligned} &\partial_y A(x, y) = 0, \ \partial_x A(x, y) + \partial_y C(x, y) = 0 \\ &\partial_x B(x, y) = 0, \ \partial_y B(x, y) + \partial_x C(x, y) = 0 \\ &bC(x, y) + 2B(x, y)v'(y) - \partial_x D(x, y) = 0 \\ &2bA(x, y) + C(x, y)v'(y) - \partial_y D(x, y) = 0 \end{aligned} \tag{23}$$

The general solution for $A(x,y)$, $B(x,y)$, $C(x,y)$

$$A(x,y) = \lambda - \gamma x - \frac{\rho x^2}{2},\ B(x,y) = \sigma - \beta y - \frac{\rho y^2}{2},\ C(x,y) = \alpha + \beta x + \gamma y + \rho x y \quad (24)$$

where α, β, γ, λ, σ, ρ are constants.

If both $\partial_x D(x,y)$ and $\partial_y D(x,y)$ are nonzero, one has the restriction

$$2(\sigma - \beta y)v''(y) - 3\beta v'(y) + 3b\gamma = 0 \quad (25)$$

The potential is not continuously differentiable everywhere.

If either $\partial_x D(x,y)$ or $\partial_y D(x,y)$ is allowed to vanish

$$b \neq 0, \qquad I_2 = p_x^2 + 2by \quad (26)$$

$$b = 0, \qquad I_1 = p_x, \qquad I_2 = p_x(xp_x - yp_y) - yv(y) + \int_0^y v(r)dr \quad (27)$$

The first case is integrable. The second case is maximally superintegrable.

Self–dual metrics with maximally superintegrable geodesic flows

The metric

$$d\tau^2 = -2v(y)dt^2 + 2dtds + 2dxdy \tag{28}$$

Killing vectors

$$K_1 = \partial_t, \qquad K_2 = \partial_s, \qquad K_3 = \partial_x \tag{29}$$

$$K_4 = t\partial_x - y\partial_s, \qquad K_5 = -y\partial_t + s\partial_x - 2\left(\int\limits_0^y v(r)dr\right)\partial_s \tag{30}$$

The second rank Killing tensor

$$K_{tt} = -yv(y) + \int\limits_0^y v(r)dr, \qquad K_{xy} = -\frac{y}{2}, \qquad K_{yy} = x \tag{31}$$

Together with $g_{AB}(z)\frac{dz^A}{d\tau}\frac{dz^B}{d\tau}$ they provide seven functionally independent integrals of motion which render the geodesic flow maximally superintegrable.

Conlusion

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Open problems

- Applications within the context of the $N = 2$ string theory
- Construction of integrable models with higher order invariants governed by a harmonic potential

Thanks for your attention!