Usage Power Geometry and Normal Form Methods for nonlinear ODEs study

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Introduction

We consider an autonomous system of ordinary differential equations of the form

$$dx_i/dt \stackrel{\text{def}}{=} \dot{x}_i = \varphi_i(X), \quad i = 1, 2 \ , \tag{1}$$

where $X = (x_1, x_2) \in \mathbb{C}^2$ and $\varphi_i(X)$ are polynomials.

The functions φ_i can be depended on parameters.

A method of the analysis of integrability of system (1) based on power transformations [Bruno: 1998] and computation of normal forms near stationary solutions of transformed systems (see [Bruno: 1971] and Ch.II in [Bruno: 1979]) was proposed in [Bruno, Edneral: 2009; Bruno, Edneral: 2013].

Plane nonlinear systems

Vibration problems with parameters, stability conditions:

- Duffing equation, blocking generator
- Predator-pray model
- Autocenter of turbines
- Astrodynamics
- A lot of other mechanical tasks
- Electric circus e.t.c.

Solutions

- Numerical solutions
- Approximated solutions
- Exact solutions

Exact solutions

- Are very rare
- Consist full information (bifurcation points, domains of stability e.t.c.)
- Allow to create approximated solution near exact solution (violation theory e.t.c.)

Under what conditions are there exact solutions?

- Let us search the values of parameters the system is integrable?
- Then we try to find the first integral of motion
- If we now these integrals we know the solutions

We consider an autonomous system of ordinary differential equations of the form

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where $X = (x_1, x_2) \in \mathbb{C}^2$ and $\varphi_i(X)$ are polynomials.

The functions φ_{i} can be depended on parameters.

At which values of the parameters are there exact solutions? I.e. at which values of parameters the system is integrable?

In a neighborhood of the stationary point $X = 0$ system (1) can be written in the form

$$\dot{X} = A\,X + \Phi(X),$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of the matrix A. If at least one of them $\lambda_i \neq 0$, then the stationary point $X = 0$ is called an *elementary* stationary point. In this case the system (2) has a normal form which is equivalent to a system of lower order [Bruno:1979]. If all eigenvalues vanish, then the stationary point $X = 0$ is called a *nonelementary* stationary point. In this case there is no normal form for the system. But by using power transformations, a nonelementary stationary point $X = 0$ can *be blown up* to a set of elementary stationary points.

After that it is possible to compute the normal form and verify that the condition **A** [Bruno:1971, 1979] (see later) is satisfied in each elementary stationary point.

In this paper we demonstrate how this approach can be applied to study the local and global integrability in the planar case of the system (1) near the stationary point $X^0 = 0$ of high degeneracy

$$\begin{aligned} \dot{x} &= \alpha\, y^3 + \beta\, x^3\, y + (a_0\, x^5 + a_1\, x^2 y^2) + (a_2\, x^4\, y + a_3\, x\, y^3), \\ \dot{y} &= \gamma\, x^2\, y^2 + \delta\, x^5 + (b_0\, x^4 y + b_1\, x\, y^3) + (b_2\, x^6 + b_3\, x^3\, y^2 + b_4\, y^4). \end{aligned} \tag{M}$$

The integrability problem for a class of planar systems was studied in [A. Algaba, E. Gamero, C. Garcia: Nonlinearity 22 (2009) 395-420].

There the authors set $-\alpha = \delta = 1$ and $3\beta + 2\gamma = 0$. Further the authors of studied the Hamiltonian case of this system with the additional assumption that the Hamiltonian function is expandable into the product of only square-free factors.

$$dx / dt = x(-x^{-1}y^{3} - bx^{2}y + a_0 x^{4} + a_1 xy^{2})$$
$$dy / dt = y((1/b)x^{2}y + x^{5}y^{-1} + b_0 x^{4} + b_1 xy^{2})$$

Local and global problems

- Local problems are research in the small neighbourhood of a some point. Different series is a very spread method of local analysis.
- Global problems are research in some domain of the phase space.
- To connect local and global approaches are my dream.
- Gustavson’s integral

We start from the study the case when the first quasi-homogeneous approximation of (M) is

$$\dot{\tilde{x}} = \alpha\,\tilde{y}^3 + \beta\,\tilde{x}^3\,\tilde{y}, \quad \dot{\tilde{y}} = \gamma\,\tilde{x}^2\,\tilde{y}^2 + \delta\,\tilde{x}^5 \ ,$$

where $\alpha \neq 0$ and $\delta \neq 0$. Using the linear transformation $x = \sigma\tilde{x}$, $y = \tau\tilde{y}$ we can fix two nonzero parameters in (H)

$$\dot{x} = -y^3 - b\,x^3\,y, \quad \dot{y} = c\,x^2\,y^2 + x^5 \ . \qquad \text{(H)}$$

Each autonomous planar quasi-homogeneous system (H) has an integral, but it can be not an analytic form. We are interested to have the local integrability of (H). It is necessary for the integrability of (M).

If you know the first integral of motion of a planar system then you know its solution.

Theorem 1 *If* b^2*=2/3 or in the case* $D \overset{\text{def}}{=} (3b + 2c)^2 - 24 \neq 0$, *system (H) is locally integrable if and only if the number* $(3b - 2c)/\sqrt{D}$ *is rational.*

In this paper we will study simple partial case where D is chosen in such way that $c=1/b$. In view of Theorem 1, the first quasi-homogeneous approximation has an analytic integral but it is not a Hamiltonian system. We will study the integrability problem for entire system (S) with the first quasi-homogeneous approximation (H)

$$\begin{aligned} dx/dt &= -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2 \;, \\ dy/dt &= (1/b)\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3 \;, \end{aligned} \tag{S}$$

So we consider the 5 parameters system with b ≠ 0.

The rationality of the ratio λ_1/λ_2 and the condition **A** (see [Bruno:1971,1979]) are necessary and sucient conditions for local analytical integrability of a planar system near an elementary stationary point.

The condition **A** is a strong algebraic condition on coefficients of the normal form. It consists of infinite numbers of algebraic equations on parameters.

For a local integrability of original system (1) near a degenerate (nonelementary) stationary point, it is necessary to have local integrability near each of elementary stationary points, which are produced by the blowing up process described below.

The algorithm for calculation of the normal form, and of the normalizing transformation together with the corresponding computer program are briefly described in [Edneral:2007].

About Resonant Normal Form and the Condition **A**

Let the linear transformation

$$X = BY \tag{11}$$

bring the matrix A to the Jordan form $J = B^{-1}AB$ and (2) to

$$\dot{Y} = JY + \tilde{\tilde{\Phi}}(Y). \tag{12}$$

Let the formal change of coordinates

$$Y = Z + \Xi(Z), \tag{13}$$

where $\Xi = (\xi_1, \ldots, \xi_n)$ and $\xi_j(Z)$ are formal power series, transform (12) in the system

$$\dot{Z} = JZ + \Psi(Z). \tag{14}$$

We write it in the form

$$\dot{z}_j = z_j g_i(Z) = z_j \sum g_{jQ} Z^Q \quad \text{over} \quad Q \in \mathbb{N}_j,\ j = 1, \ldots, n, \tag{15}$$

where $Q = (q_1, \ldots, q_n)$, $Z^Q = z_1^{q_1} \ldots z_n^{q_n}$,

$$\mathbb{N}_j = \{Q : Q \in \mathbb{Z}^n,\ \ Q + E_j \geq 0\},\ j = 1, \ldots, n,$$

E_j means the unit vector. Denote

$$\mathbb{N} = \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_n. \tag{16}$$

The diagonal $\Lambda = (\lambda_1, \ldots, \lambda_n)$ of J consists of eigenvalues of the matrix A.

System (14), (15) is called the *resonant normal form* if:

a) J is the Jordan matrix,

b) in writing (15), there are only the *resonant terms*, for which the scalar product

$$\langle Q, \Lambda \rangle \stackrel{\text{def}}{=} q_1\lambda_1 + \ldots + q_n\lambda_n = 0. \tag{17}$$

Theorem 1 (Bruno [4]) *There exists a formal change (13) reducing (12) to its normal form (14), (15).*

In [Bruno:1971] was proved that there are conditions on the normal form (15), which guarantee the convergence of the normalizing transformation (13).

Condition A. *In the normal form (15)*

$$g_j = \lambda_j \alpha(Z) + \bar{\lambda}_j \beta(Z), \quad j = 1, \dots, n,$$

where $\alpha(Z)$ and $\beta(Z)$ are some power series.

Let

$$\omega_k = \min |\langle Q, \Lambda \rangle| \ \text{ over } \ Q \in \mathbb{N}, \ \ \langle Q, \Lambda \rangle \neq 0, \ \ \sum_{j=1}^{n} q_j < 2^k, \ \ k = 1, 2, \dots$$

Condition ω (on small divisors). *The series*

$$\sum_{k=1}^{\infty} 2^{-k} \log \omega_k > -\infty,$$

i.e. it converges.

It is fulfilled for almost all vectors Λ.

Theorem 2. Bruno 1971.*If vector Λ satisfies Condition ω and the normal form (2.6) satisfies Condition A then the normalizing transformation (13) converges*

The algorithm of a calculation of the normal form, the normalizing transformation and the corresponding computer program are described in [Edneral:2007].

Power Transformation

Let

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

be a matrix with real elements and $\det \alpha \neq 0$. Then the *power transformation*

$$y_1 = x_1^{\alpha_{11}} x_2^{\alpha_{12}} ,$$

$$y_2 = x_1^{\alpha_{21}} x_2^{\alpha_{22}}$$

has the inverse

$$x_1 = y_1^{\beta_{11}} y_2^{\beta_{12}} , \qquad \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} ,$$

$$x_2 = y_1^{\beta_{21}} y_2^{\beta_{22}} ,$$

where $\beta = \alpha^{-1}$. In fact, transformations above are linear with respect to the logarithms of the coordinates:

$$\begin{cases} \ln y_1 = \alpha_{11} \ln x_1 + \alpha_{12} \ln x_2 , \\ \ln y_2 = \alpha_{21} \ln x_1 + \alpha_{22} \ln x_2 ; \end{cases}$$

$$\begin{cases} \ln x_1 = \beta_{11} \ln y_1 + \beta_{12} \ln y_2 , \\ \ln x_2 = \beta_{21} \ln y_1 + \beta_{22} \ln y_2 . \end{cases}$$

Blow-Up procedure

$$\vec{Q} = Q_2 - Q_1 \equiv \begin{pmatrix} q_2 \\ q_1 \end{pmatrix},$$

$$\vec{Q}` = Q_2` - Q_1` \equiv \begin{pmatrix} q_2` \\ 0 \end{pmatrix},$$

$$\vec{Q}` = \hat{\alpha}\,\vec{Q}; \hat{\alpha} = ?,$$

$$\vec{P} = (-q_1, q_2),$$

$$< \vec{P}, \vec{Q} >= 0.$$

$$\hat{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ -q_1 & q_2 \end{pmatrix},$$

$\det \hat{\alpha} = \pm 1,$ It is suitable but a not necessary rule.

$q_2\alpha_{11} - q_1\alpha_{11} = 1,$ The upper elements can be find by extended Euclid's algorithm.

$\hat{\alpha}^{+}$ Really we should use conjugated matrix.

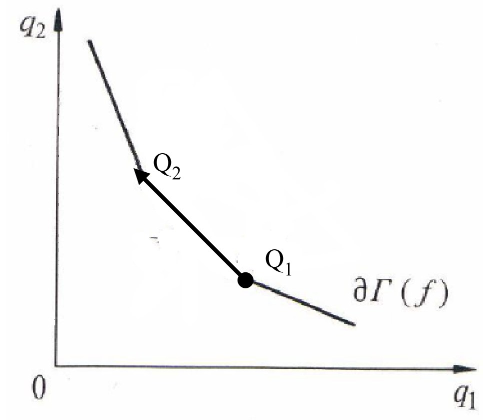


Fig. 1

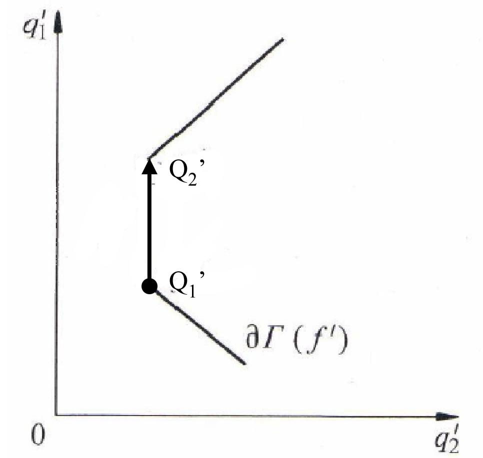


Fig. 2

This is the simplest nontrivial quasi-homogeneous

5 parametric example

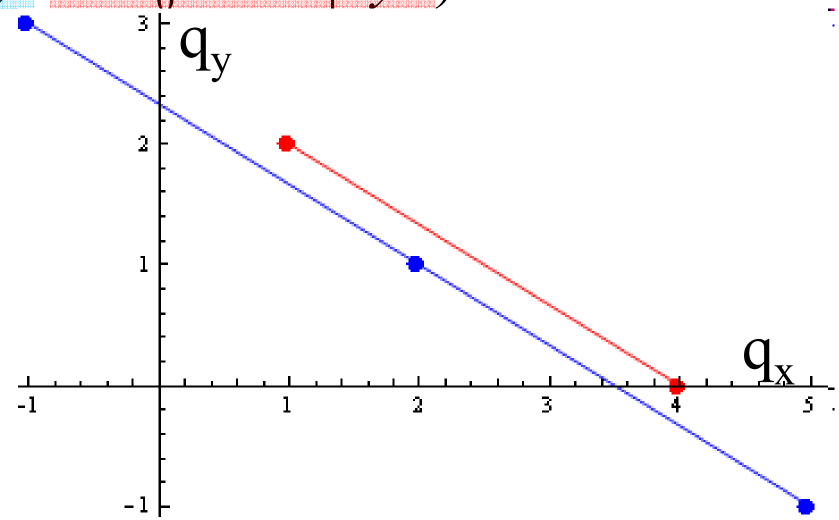
We consider the system

$$\begin{aligned} dx/dt &= -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2, \\ dy/dt &= (1/b)\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3, \end{aligned} \qquad (18)$$

with arbitrary complex parameters a_i, b_i and $b \neq 0$.

$$dx/dt = x(-x^{-1}y^3 - bx^2y + a_0x^4 + a_1xy^2)$$

$$dy/dt = y((1/b)x^2y + x^5y^{-1} + b_0x^4 + b_1xy^2)$$



$$\vec{Q} = Q_2 - Q_1 \equiv \begin{pmatrix} -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix},$$

$$\vec{P} = (2,3),$$

$$< \vec{P}, \vec{Q} >= 0.$$

$$\hat{\alpha} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix},$$

$$\hat{\alpha}^{+} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} \log x \\ \log y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \log u \\ \log v \end{pmatrix}.$$

At the first step we should rewrite (S) in a non-degenerate form. It can be realized by the power transformation [Bruno:1998, BrunoEdneral:2009]

$$x = u\, v^2, \quad y = u\, v^3.$$

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{u'[t] == 1/b u[t]^3 v[t]^7 (-3 b + u[t] (-2 - 3 b^2 - 2 b u[t] + b (3 a1 - 2 b1 + (3 a0 - 2 b0) u[t]) v[t])),
 v'[t] == (u[t]^2 v[t]^8 (b + u[t] (1 + b^2 + b u[t] + b (-a1 + b1 + (-a0 + b0) u[t]) v[t])))/b}
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With time rescaling $u2\ v7\ d\,t = d\,\tau$ we obtain the system (S) in the form

$$\begin{aligned} du/d\tau &= -3\,u - [3\,b + (2/b)]u^2 - 2\,u^3 + (3\,a_1 - 2\,b_1)u^2 v + (3\,a_0 - 2\,b_0)u^3 v\ , \\ dv/d\tau &= v + [b + (1/b)]u\,v + u^2 v + (b_1 - a_1)u\,v^2 + (b_0 - a_0)u^2 v^2\ . \end{aligned} \tag{T1}$$

Under the power transformation above the point $x = y = 0$ blows up into two straight invariant lines: $u = 0$ and $v = 0$. Along the line $u = 0$ the system has two stationary point $u = v = 0$ and $u = 0$, $v = \infty$. Along the second line $v = 0$ this system has four elementary stationary points

$$u = 0, \quad u = -\frac{1}{b}, \quad u = -\frac{3b}{2}, \quad u = \infty\ .$$

Lemma 1 *Near the points* $u = v = 0$; $u = 0$, $v = \infty$ *and* $u = \infty$, $v = 0$ *system* (T1) *is locally integrable.*

Thus we must find conditions of local integrability at two other stationary points

$$u = -\frac{1}{b}, \quad u = -\frac{3b}{2},$$

then we will have the conditions of local integrability of the system (T1) near the origin point.

Let us consider the stationary point $u = -1/b$; $v = 0$. Firstly we restrict ourselves to the case $b^2 \neq 2/3$ when the linear part of system (T1), after the shift $u = w - 1/b$, has non-vanish eigenvalues. At the subcase $b^2 = 2/3$ the matrix of the linear part of the shifted system in new variables w and v has Jordan cell with both zero eigenvalues (see T2 later). This case will be studied by means of one more power transformation below.

We computed the condition **A** with program [Edneral:2007]. There are 2 solutions of a corresponding subset of equations from the condition **A** at b $\neq 0$

$$a_0 = 0, \quad a_1 = -b_0\, b, \quad b_1 = 0, \quad b^2 \neq 2/3$$

and

$$a_0 = a_1\, b, \quad b_0 = b_1\, b, \quad b^2 \neq 2/3 \ .$$

The consideration of the stationary point u = -3 b/2, v = 0 under the last condition from above gives tree more two-parameters (a_1 and b) solutions

$$\begin{array}{llll} 1) \quad b_1 = -2\, a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3 \ , \\ 2) \quad b_1 = (3/2)\, a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3 \ , \\ 3) \quad b_1 = (8/3)\, a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3 \ . \end{array}$$

Theorem 2. *Conditions belov form a set of necessary conditions of a local integrability of the system* (T1) *in all its stationary points and a local integrability of system* (S) *at stationary point* $x = y = 0$.

1. $a_0 = 0, \quad a_1 = -b_0\, b, \quad b_1 = 0, \quad b^2 \neq 2/3$
2. $b_1 = -2\, a_1, \qquad a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3$,
3. $b_1 = (3/2)\, a_1, \quad a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3$,
4. $b_1 = (8/3)\, a_1, \quad a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3$.

Sufficient conditions of global integrability

The conditions presented in theorem 2 are necessary and sucient for local integrability of system (S) in the zero stationary point. They can be considered as good candidates for sucient conditions of the global integrability. However it is necessary to prove the sufficiency of these conditions by independent methods. It is necessary to do it for each of four conditions above.

In [EdneralRomanovski:2010] we found first integrals for all these cases mainly by the Darboux factor method for the system (S).

Darboux's method

$$\begin{aligned} \dot{u} &= -u(9b + 6u + 7a_1buv) \\ \dot{v} &= v(3b + 3u + 5a_1buv). \end{aligned} \qquad \textbf{(U)}$$

The sense of the integrating factor M is that consequence of equations (U)

$$M(u,v)u(9b + 6u + 7a_1buv)dv/du + M(u,v)v(3b + 3u + 5a_1buv) = 0$$

will be an equation in full differentials. It means that there is the function $F(u,v)$ which has the continuous partial derivatives. It means that

$$F_u(u,v) = M(u,v)v(3b + 3u + 5a_1buv),$$
$$F_v(u,v) = M(u,v)u(9b + 6u + 7a_1buv).$$

$$M = \frac{1}{u^{4/3}v^2\left[6u + b(3 + a_1uv)^2\right]^{7/6}}.$$

And solutions of (U) will have the form $v = \varphi(u)$, for which

$$F(u, \varphi(u)) = const.$$

So, $F(u,v)$ is the first integral of motion.

1. At $a_0 = 0, \quad a_1 = -b_0\, b, \quad b_1 = 0$:

$$I_{1uv} = u^2(3\,b + 2\,u)v^6 \;,$$
$$I_{1xy} = 2\,x^3 + 3\,b\,y^2 \;.$$

2. At $b_1 = -2a_1,\, a_0 = a_1 b,\, b_0 = b_1 b$:

$$I_{2uv} = u^2\, v^6\,(3\,b + u\,(2 - 6\,a_1\,b\,v)) \;,$$
$$I_{2xy} = 2\,x^3 - 6\,a_1\,b\,x^2\,y + 3\,b\,y^2 \;.$$

3. At $b_1 = 3a_1/2,\, a_0 = a_1 b,\, b_0 = b_1 b$:

$$I_{3uv} = [4 - 4a_1\,u\,v + 3^{5/6}a_1 \times {}_2F_1\,(2/3, 1/6; 5/3; -2u/(3b))\, u\,v\,(3 + 2u/b)^{1/6}]/[u^{1/3}v\,(3b + 2u)^{1/6}] \;,$$

$$I_{3xy} = [a_1 x^2(-4 + 3^{5/6}\;{}_2F_1\,(2/3, 1/6; 5/3; -2\,x^3/(3\,b\,y^2)) \times (3 + 2x^3/(b\,y^2))^{1/6}) + 4y]/[y^{4/3}(3\,b + 2\,x^3/y^2)^{1/6}] \;,$$

4. At $b_1 = 8a_1/3,\, a_0 = a_1 b,\, b_0 = b_1 b$:

$$I_{4u,v} = [u\,(3 + 2\,a_1^2 bu) + 6\,a_1\,b\,v]/ [3\,u\,[u^3(6 + a_1^2 b\,u) + 6\,a_1^2 b\,u^2 v + 9\,b\,v^2]^{1/6}] - 8\,a_1\sqrt{-b}/3^{5/3} B_{6 + a_1\sqrt{-6\,b\,u} + 3\,v\,\sqrt{-6\,b/u^3}}(5/6, 5/6) \;,$$

where the B_y(a,b) is the incomplete beta function and ${}_2F_1$(a,b;c;z) is a hypergeometric function [Bateman:1953].

The integrals (and solutions) do not have any singularities near the points $b^2 = 2/3$, but the approach in which these solutions were found has the limitation $b^2 \neq 2/3$, so there are possible additional solutions at this point. Thus we need to study this case separately.

Case $b^2 = 2/3$,
Subcase $3a_0 - 2b_0 = b(3a_1 - 2b_1)$

At values $b^2 = 2/3$ the both stationary points $u = -3b/2$, v = 0 and $u = -1/b$, v =0 are collapsing and after the shift u → w -1/b we have instead of (T1) the nilpotent degenerated system

$$\begin{aligned}
\frac{dw}{d\tau} &= -3v/(2b)[(3a_0 - 2b_0) - b(3a_1 - 2b_1)] + \\
& wv(\tfrac{27}{2}\,a_0 - 3\sqrt{6}\,a_1 - 9\,b_0 + 2\sqrt{6}\,b_1) + \\
& \sqrt{6}\,w^2 + w^2 v(-9\sqrt{\tfrac{3}{2}}\,a_0 + 3\,a_1 + 3\sqrt{6}\,b_0 - 2\,b_1) - \\
& 2\,w^3 + w^3 v(3\,a_0 - 2\,b_0)\ , \\
\frac{dv}{d\tau} &= -\frac{\sqrt{6}}{6}wv + v^2(-\tfrac{3}{2}\,a_0 + \sqrt{\tfrac{3}{2}}\,a_1 + \tfrac{3}{2}\,b_0 - \sqrt{\tfrac{3}{2}}\,b_1) + \\
& w^2 v + wv^2((\sqrt{6}\,a_0 - a_1 - \sqrt{6}\,b_0 + b_1) + \\
& + w^2 v^2(-a_0 + b_0)\ .
\end{aligned} \qquad \text{(T2)}$$

So we should apply a power transformation once again.

$$\vec{Q} = Q_2 - Q_1 \equiv \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

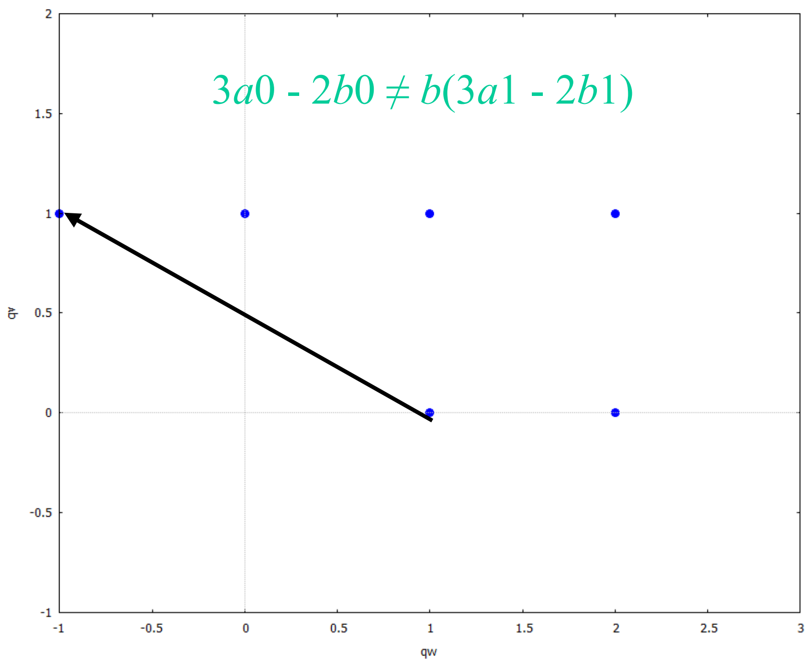
$$\vec{P} = (1,2), < \vec{P}, \vec{Q} >= 0.$$

$$\hat{\alpha}^{+} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} \log s \\ \log v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \log w \\ \log r \end{pmatrix},$$

$s = wr, v = wr^2$. It gives a trivial result.

$$\hat{\alpha}^{+} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} \log s \\ \log v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \log w \\ \log r \end{pmatrix},$$

$w = w, v = wr^2$.



In the paper [BrunoEdneral:2013] we used the last transformation and got the systems with resonances of 19th and 27th orders. We calculated the corresponding normal form with 4 free parameters till 19th order but for finding new solutions of the condition **A** we need to calculate normal form till 27th order. The last resonance exists if $b^2 = 2/3$, $3a_0 - 2b_0 \neq b(3a_1 - 2b_1)$ only and its calculation is very hard. We postpone this investigation and consider here the other partial subcase when $3a_0 - 2b_0 = b(3a_1 - 2b_1)$, $b^2 = 2/3$.

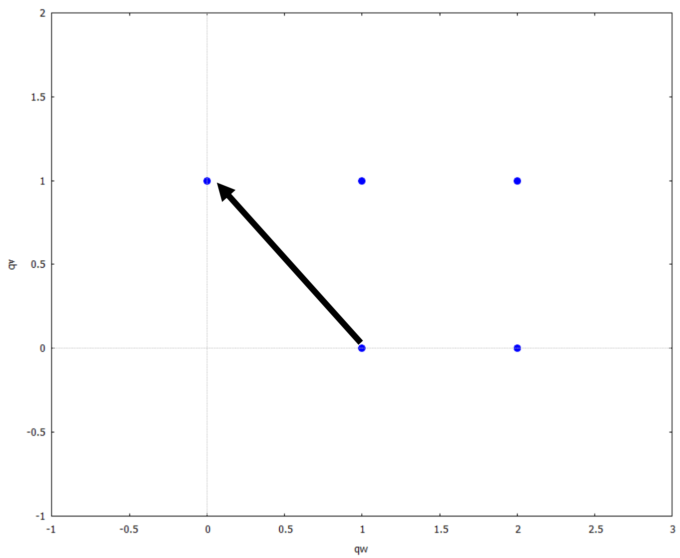
We see that in the system above, the coefficient of v in the linear part of the first equation is zero if $3a_0 - 2b_0 = b(3a_1 - 2b_1)$. So we have the special subcase.

$$\vec{Q} = Q_2 - Q_1 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$$\vec{P} = (1,1), < \vec{P}, \vec{Q} >= 0.$$

$$\hat{\alpha}^{+} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \log v \\ \log w \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \log r \\ \log w \end{pmatrix},$$

$$w = w, v = wr.$$



So, for this subcase we use the transformation

$$u = w - 1/b, v = wr.$$

We then have from (T1)

$$\begin{aligned}
\frac{dr}{d\tilde{\tau}_1} &= -7r - \left(9a_1 - \sqrt{\tfrac{3}{2}}b_0 - 5b_1\right)r^2 + 3\sqrt{6}rw + \\
&\quad \left(7\sqrt{6}a_1 - 2b_0 - 13\sqrt{\tfrac{2}{3}}b_1\right)r^2 w - \left(8a_1 - \sqrt{\tfrac{3}{2}}b_0 - \tfrac{16}{3}b_1\right)r^2 w^2 \ , \\
\frac{dw}{d\tilde{\tau}_1} &= 6w + 3(3a_1 - 2b_1)rw - 2\sqrt{6}w^2 - 2\sqrt{6}(3a_1 - 2b_1)rw^2 + \\
&\quad 2(3a_1 - 2b_1)rw^3 \ .
\end{aligned} \qquad \text{(T3)}$$

This is a three parameters system with a resonance of the 13th order at the stationary point $r = 0$, $w = 0$ on the invariant line $w = 0$. At this it is also another stationary point and one more point in infinity but they have non-rational quotient of eigenvalues, so they are integrable under condition **A**.

We calculated the normal form for (T3) at $r = 0$, $w = 0$ till the 26th order and got two equations for the condition **A**. They are a13 = 0 and a26 = 0 where a13 and a26 are homogeneous polynomials in parameters a_1, b_0, b_1 of system (S) sixth and twelfth orders correspondingly.

For example a13 is

a13 =
77591416320*a1^6*s6+65110407552*a1^5*b0-
343384549344*a1^5*b1*s6-
214574033664*a1^4*b0^2*s6-1084658542848*a1^4*b0*b1+
495240044652*a1^4*b1^2*s6-618953467392*a1^3*b0^3+
59995851552*a1^3*b0^2*b1*s6+1782026653968*a1^3*b0*b1^2-
325584668628*a1^3*b1^3*s6-8037029376*a1^2*b0^4*s6+
642627782784*a1^2*b0^3*b1+230489977896*a1^2*b0^2*b1^2*s6-
1080958485096*a1^2*b0*b1^3+105084809187*a1^2*b1^4*s6-
29504936448*a1*b0^5+95627128896*a1*b0^4*b1*s6-
130189857408*a1*b0^3*b1^2-155744503512*a1*b0^2*b1^3*s6+
270984738720*a1*b0*b1^4-15802409798*a1*b1^5*s6+
19669957632*b0^5*b1-20179406208*b0^4*b1^2*s6-
15425489664*b0^3*b1^3+25998124528*b0^2*b1^4*s6-
22559067296*b0*b1^5+882415736*b1^6*s6,

where $s6 = \sqrt{(6)}$. Both a13 and a26 are equal to zero at the founded before solutions.

Homogeneous algebraic equations in three variables can be rewritten as inhomogeneous equations in two variables.

If we suppose that $a_1 = 0$, we get only one and zero dimensional solutions in the parametric cospace. Let us postpone the consideration of these cases and suppose that $a_1 \neq 0$. In this case, we substitute $b_0 = c_0\, a_1$; $b_1 = c_1\, a_1$ and obtain the system of two equations in two variables a13(c0; c1) = 0; a26(c0; c1) = 0. The resultant of two corresponding polynomials in each of two variables is identically equal to zero. So it is enough to solve equation a13(c0; c1) = 0 and check higher orders.

It is interesting that the condition **A** of the 19th order from [Bruno,Edneral:2013, A19] is identically equal to a13 up to multiplication by a constant at the subcase $b^2 = 2/3$, Subcase $3a_0 - 2b_0 = b(3a_1 - 2b_1)$.

Equation a13 = 0 can be factorized as the product of four factors including a_1^6:

$$
\begin{aligned}
a13 = {} & 48(c_1 - 3/2)\times \\
& (c_0 - 1/12\sqrt{6}c_1 + 1/2\sqrt{6})^2\times \\
& [409790784c_0^3 - 104\sqrt{6}c_0^2(-9152256 + 3385633c_1)- \\
& 208c_0(-10917702 + c_1(-360720 + 3319927c_1))+ \\
& \sqrt{6}(-718439040 + c_1(2461047528+ \\
& c_1(-1944898681 + 441207868c_1)))]\times \\
& a_1^6 \; .
\end{aligned}
$$

From the first two factors we get two of two-parametric solutions

$$\begin{array}{lll} b_1 = 3/2, & a_0 = (2b_0 + b(3a1 - 2b1))/3, & b = \sqrt{2/3}\ , \\ b_1 = 6a_1 + 2\sqrt{6}b_0, & a_0 = (2b_0 + b(3a1 - 2b1))/3, & b = \sqrt{2/3}\ . \end{array} \quad \text{(NP)}$$

For these solutions we calculate the normal form of (T3) till the 36th order. And for each solution it is a diagonal linear system.

The research of the cubic factor above is very hard. But fortunately the resultant of this cubic factor with A26 is a polynomial in c_0 or c_1. It has a finit numbers of solution. The cubic factor will have a finit numbers corresponding zeroes also. So it can not give any additional two-dimension solutions.

New integrals of motion

For each set of parameters (NP) one can find Darbouxs integration factor $\mu = f_1^{a} \cdot f_2^{d} \cdot f_3$, see [Romanovski,Shafer:2009]. In both cases system (T3) has invariant lines $f_1 = r$, $f_2 = w$, $f_3 = 1 - \sqrt{2/3}w$.

In the first case (when $b_1 = 3/2a_1$)

$$\mu_1 = r^a w^d f_3^c \ ,$$

where

$$a = -2, \quad d = -\frac{13}{6}, \quad c = -\frac{4}{3} \ .$$

In the second case (when $b_1 = 6a_1 + 2\sqrt{6}b_0$)

$$\mu_2 = r^a w^d f_3^c \ ,$$

where

$$a = \frac{3a_1 + 2\sqrt{6}b_0}{3a_1 + \sqrt{6}b_0}, \quad d = \frac{8a_1 + 5\sqrt{6}b_0}{6a_1 + 2\sqrt{6}b_0}, \quad c = \frac{-a_1}{3a_1 + \sqrt{6}b_0} \ .$$

The corresponding first integrals of equations (T3) are

$$I_{1rw} = w^{-7/6}(1-\sqrt{\tfrac{2}{3}}w)^{-1/3}[-9a_1+3\sqrt{6}b_0-\tfrac{42}{r}-6(\sqrt{6}a_1+5b_0)w+ \\ 2(9a_1+4\sqrt{6}b_0)w^2-2^{1/6}(9\sqrt{2}a_1+8\sqrt{3}b_0)w^{5/3}(-\sqrt{6}+2w)^{1/3}\cdot \\ {}_2F_1(-1/2,1/3;1/2;\sqrt{2/3}/w)]\ ,$$

$$I_{2rw} = r^{3\frac{3a_1}{3a_1+\sqrt{6}b_0}}\cdot w^{7/3+\frac{7b_0}{3\sqrt{6}a_1+6b_0}}\cdot(1-\sqrt{2/3}w)^{\frac{-a_1}{3a_1+\sqrt{6}b_0}}\cdot \\ \{\tfrac{-6+2\sqrt{6}w}{6a_1+3\sqrt{6}b_0}+r[3+2w(-\sqrt{6}+w)]\}\ .$$

In the origin variables x; y corresponding integrals of equations (S) have the form accurate to numerical factor

$$
\begin{aligned}
I_{1xy} = {} & (y/x^2)(\sqrt{6} + 2x^3/y^2)^{-7/6}(x^3/y^2)^{2/3} \cdot \{42\sqrt{6} + \\
& 1/(xy^3)[-36a_1x^6 - 16\sqrt{6}b_0x^6 + 84x^4y24\sqrt{6}a_1x^3y^2 - 36b_0x^3y^2 + \\
& 2^{1/3}(x^3/y^2)^{1/3}y^2(\sqrt{6} + (x^3/y^2)^{2/3} \cdot \\
& (2(9a_1 + 4\sqrt{6}b_0)x^3 + 3(3\sqrt{6}a_1 + 8b_0)y^2) \cdot \\
& {}_2F_1(-1/2, 1/3; 1/2; \tfrac{3y^2}{3y^2 + \sqrt{6}x^3})]\} \ ,
\end{aligned}
$$

$$
\begin{aligned}
I_{2xy} = {} & y(\sqrt{2/3} + x^3/y^2)^{-1/2 + \frac{a_1}{-6a_1 - 2\sqrt{6}b_0}}(x^2/y)^{-\frac{a_1}{3a_1 + \sqrt{6}b_0}} \cdot \\
& \{3 + (x^2/y^2)[\sqrt{6}x + 3(2a_1 + \sqrt{6}b_0)y]\} \ .
\end{aligned}
$$

Analytical Properties of the Integrals

We should check analyticity of the obtained first integrals near the origin$x = y = 0$. We note that by Theorem 4.13 of [Christopher, Mardesic, Rousseau: 2003] if a system has a Darboux integrating factor of the form

$$\mu = f_1^{\beta_1} f_2^{\beta_2} (1 + \mathrm{h.o.t})^{\beta}$$

then it has an analytic rst integral except of the case when both β_1 and β_2 are integer numbers greater than 1. In the both cases above orders a and b of the integrating factor $\mu_{1,2}$ are not integer simultaneously in general position. It appears integrals are not analytic, but by the theorem mentioned above the system has also local analytic first integrals, which may be difficult to obtain in a closed form.

Conclusions

For a five-parameter non-Hamiltonian planar system, we have found for the case $b^2 \neq 2/3$ four sets of two-parametric necessary conditions on parameters under which the system is locally integrable near the degenerate point $x = y = 0$. These sets of conditions are also sucient for local and global integrability of system (6). For the subcase $b^2 = 2/3$ and $3a_0$-$2b_0 = b(3a_1$-$2b_1)$, we have found two more first integrals. For the further search of additional first integrals, we need to calculate the condition **A** at the point with the resonance of the 27th order for the subcase $b^2 = 2/3$, $3a_0 - 2b_0 \neq b(3a_1 - 2b_1)$ [Bruno,Edneral:2013].

We have used Standard Lisp for the normal forms calculations. The integrating factors and integrals were calculated using the computer algebra system Mathematica.

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Thanks for your attention