Three-loop massive effective potential from differential equations

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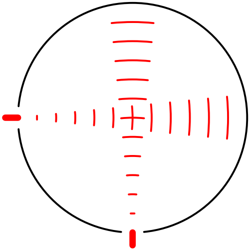
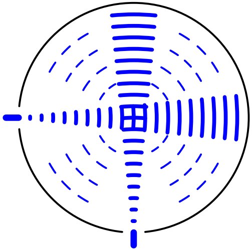
April 26, 2018

Effective theories

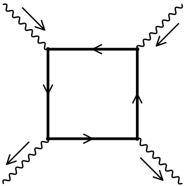
- Example from QED: [Grozin'09]





$$\mathcal{L} = \quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad + \quad c_1\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + c_2 F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}$$

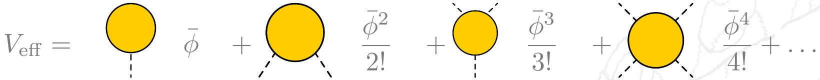
- But also we have interecting fermions term $\bar{\psi}\gamma^\mu\psi A_\mu$



In the $m_e \to \infty$ limit lead to the massive tadpoles and from them we define c_1 and c_2

Effective approach to the scalar potential

- Near phase transiotion point physics become very sensitive to input parameters and higher order effects e.g.
  - Phase transitions in various condensed matter systems described different varioations of the $\mathcal{O}(N)$ φ^4 theories
  - Standard Model near the point of the spontaneous symmetry breaking
- Effectinve potential approach is a way to account infinite sum of the higher dimensional interaction terms in lagrangian with higher order perturbative corrections



- From the QED example we saw how to account for the fixed dimensionality terms, but what about **all** posible cases?

Lagrangian parameters and mass scales

$$\mathcal{L}_S = \underbrace{\frac{{m_H}^2}{2}H^2 + \frac{{m_G}^2}{2}G_i^2}_{\text{mass terms}} + \underbrace{\frac{\tau_0}{6}H^3 + \frac{\tau_i}{6}HG_i^2}_{\text{triple interaction}} + \underbrace{\frac{\lambda_0}{24}H^4 + \frac{\lambda_i}{12}H^2G_i^2 + \frac{\lambda_{ij}}{24}G_i^2G_j^2}_{\text{quartic interaction}}$$

- $O(N)$ symmetric scalar φ^4 theory with $\langle\varphi_1\rangle = v \neq 0$ and all other $\langle\varphi_i\rangle = 0$

$$\mathcal{L} = \frac{m^2}{2}\varphi^2 + \frac{\lambda}{24}(\varphi^2)^2$$

$$\tau_0 = \tau_i = \lambda v, \quad \lambda_0 = \lambda_i = \lambda_{ij} = \lambda \quad m_H^2 = m^2 + \frac{\lambda}{2}v^2, m_G^2 = m^2 + \frac{\lambda}{6}v^2$$

- Standard Model in the broken phase

$$\mathcal{L} = m^2\Phi^\dagger\Phi + \frac{\lambda}{6}(\Phi^\dagger\Phi)^2, \quad \Phi = \frac{1}{\sqrt{2}}(v + H + iG_0, G_r + iG_i)^T$$

$$\tau_0 = \tau_i = \lambda v, \quad \lambda_0 = \lambda_i = \lambda_{ij} = \lambda \quad m_H^2 = m^2 + \frac{\lambda}{2}v^2, m_G^2 = m^2 + \frac{\lambda}{6}v^2$$

Known results for the effective potential

2-loop Analytically SM [Ford,Jack,Jones'93], general theory [Martin'01]

3-loop Numerically general theory [Martin'17]

$$D(m_H) = \text{———} = \frac{1}{p^2 + m^2 + \frac{\lambda v^2}{2}}, \quad D(m_G) = \text{- - - - -} = \frac{1}{p^2 + m^2 + \frac{\lambda v^2}{6}}$$

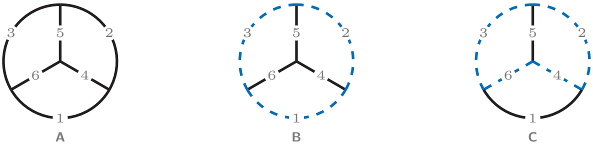
Three-loop analytically known results:

- Massless broken $\mathcal{O}(N)$ symmetric φ^4 theory (only $\mathcal{O}(\frac{1}{\varepsilon})$ part) $m_G^2 = 3m_H^2$
  [Chung,Chung'97;Kotikov'98]
- Single component masive φ^4 theory only m_H
  [Chung,Chung'99]

Our goal:

General case $m \neq 0$ with two scales $m_G \neq m_H$

Three-loop topologies for vacuum integrals



- Topology **A** is a single scale, reduction using `MATAD` [Steinhauser'00] master integrals known up to the weight 6 [Kniehl,A.P.,Veretin'17]
- Topologies **B** and **C** have 11 master integrals each depending on a single variable $x = \left(\frac{m_G}{m_H}\right)^2$, reduction using `LiteRed` [Lee'14]
- Differentiating in x and reducing back to the set of the mater integrals we obtain closed system of 11 differential equations:

$$\partial_x J_a(\varepsilon, x) = M_{ab}(\varepsilon, x) J_b(\varepsilon, x)$$

- We are looking for the solution as a series expansion in $\varepsilon = 2 - d/2$

Iterated integrals

$$\int_0^x dz_1 f_1(z_1) \int_0^{z_1} dz_2 f_2(z_2) \int_0^{z_2} dz_3 f_3(z_3) \cdots \int_0^{z_{n-1}} dz_n f_n(z_n)$$

- Harmonic polylogarithms(HPL), include Li_n and $S_{n,p}$

$$f_{-1}(z) = \frac{1}{z-1}, \quad f_0(z) = \frac{1}{z}, \quad f_1(z) = \frac{1}{z+1}$$

- Generalized polylogarithms(GPL), include HPL

$$f_a(z) = \frac{1}{z-a}$$

- Cyclotomic polylogarithms, after factorization over $\mathbb{C}$ and partial fractioning can be reexpressed through GPL

$$f_a^b(z) = \frac{z^b}{\Phi_a(z)}, f_0^0(z) = 1/z, \quad \Phi_n(z) = \prod_{\gcd(k,n)=1} \left(z - e^{2\pi i \frac{k}{n}}\right)$$

Differential equations and canonical basis

- Set of the master integrals is not unique, we are looking for the basis, which coefficients of ε-expansion have constant transcendental weight
- Differentiation reduces transcendental weight by one, if we assign weight one to ε, DE for integrals in a new basis would have following form [Henn'13]:

$$\partial_x g_a(x) = \varepsilon M_{ab}(x) g_b(x)$$

- For the coefficients of ε-expansion system decouple and solution can be written explicitly, upto a constant for each integration:

$$g_a\{\varepsilon^n\}(y) = \int dy\, M_{ab}(y)\, g_b\{\varepsilon^{n-1}\}(y) + \mathcal{C}_{a,n}$$

- For system solvable in terms of GPL, algorithmic ways of canonical basis construction exists [Lee'14] and [Meyer'16] with public implementations `Fuchsia` [Gituliar,Magerya'17], `epsilon` [Prausa'17] and `CANONICA` [Meyer'17]
- Rational transformation can be constructed only after apropriate variable change

$$x = \frac{y^2}{(1+y^2)^2}$$

Cyclotomic polylogarithms integration

- System in canonical basis can be easily decomposed into the form, where $B_{a,b}$ and $C_{a,b}$ are pure numeric matrices and all y dependence is inside functions f_a^b known how to integrate using definition of CPL:

$$\begin{aligned}
B(y) = \big(&f_0^0 B_{0,0} + f_1^0 B_{1,0} + f_2^0 B_{2,0} + f_3^0 B_{3,0} + f_3^1 B_{3,1} \\
&+ f_4^1 B_{4,1} + f_6^0 B_{6,0} + f_6^1 B_{6,1} + f_{12}^1 B_{12,1} + f_{12}^3 B_{12,3}\big) \\
C(y) = \big(&f_0^0 C_{0,0} + f_1^0 C_{1,0} + f_2^0 C_{2,0} + f_3^0 C_{3,0} + f_3^1 C_{3,1} \\
&+ f_4^1 C_{4,1} + f_6^0 C_{6,0} + f_6^1 C_{6,1} + f_8^3 C_{8,3}\big)
\end{aligned}$$

- Integration constants fixed from the finite number of terms in small m_G mass expansion ($y \to 0$) of integrals and expansion of result in terms of CPL using `HarmonicSums` package [Ablinger'13] and our own implementation
- Finite parts of the three-loop integrals are expressible through the cyclotomic polylogarithms up to the weight four

Numerical evaluation and transformations

- Cyclotomic polylogarithms are easy to evaluate with high precision as a series expansion near zero
- Comparing to HPL and even GPL lack of transformation rules like $x \to 1 - x$
- Differential equations with initial conditions are known for CPL
- Possible to construct terms of series expansion in different regions using expansion around singular points [Lee,Smirnov,Smirnov'17]

Conclusion

1. We have calculated closed three-loop analytical expression for the massive scalar theory in the broken phase in the broken phase

2. New set of the two-mass three-loop tadpole integrals calculated

3. New set of functions from the iterated integrals class used to represent results of the calculation and need further investigation

Thank you for attention!