Quantum Field Theory Methods in Classical Physics

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New Trends in High-Energy Physics

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- QFT - UV, IF divergences, RG approach
- running constants (charges)

$$Q(p'^2) = \frac{Q(p^2)}{1 - aQ(p^2)\ln(\frac{p'^2}{p^2})}$$

- a > 0 QED - corrections to the Coulomb potential
- Lamb shift
- a< 0 QCD - asymptotic freedom

Running parameters in classical systems

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} \qquad (\nabla\cdot\mathbf{v}) = 0, \qquad \mathbf{v} \equiv \mathbf{v}(\mathbf{x}, t)$$

- Hydrodynamic turbulence - example of system with strong flucuations
- Reynolds number $Re = \frac{VL}{\nu} \gg 1$
- Running parameter: turbulent viscosity

$$\nu_t(p'^2) = \nu_t(p^2)\left(\frac{p'^2}{p^2}\right)^{-\frac{2}{3}}$$

- Gell-Mann-Low equations

$$s\frac{d\bar{g}_i(s)}{ds} = \beta_i(\bar{g}(s)), \quad \bar{g}(s) \equiv \bar{g}_1(s)...\bar{g}_n(s), \quad s \equiv p/\mu, \quad \bar{g}_i(1) = g$$

QFT - generating functional

Functional formulation of QFT

- Generating functionals

$$G(A) = \frac{\int D\phi e^{S(\phi)+\phi A}}{\int D\phi e^{S(\phi)}}, \qquad \phi A \equiv \int dx \phi(x) A(x)$$

- Wick theorem

$$G(A) = e^{\frac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}} e^{S_I(\phi)+\phi A}|_{\phi=0}, \quad S(\phi) = -\frac{1}{2}\phi K \phi + S_I(\phi), \quad \Delta \equiv K^{-1}$$

- Feature: Euclidian space of coordinates, time t is singled out

A. N. Vasil'ev, The field theoretic renormalization group in critical behavior theory and stochastic dynamics, Boca Raton: Chapman Hall/CRC, 2004

QFT - Action functionals

- QED action

$$S(\psi,\bar{\psi},A_\mu) \equiv \int dx\,\mathcal{L} = \int dx\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}[-i\gamma^\mu\partial_\mu + m]\psi - e\bar{\psi}\gamma^\mu\psi A_\mu\}$$

- Static action of Ginsburg-Landau fluctuating φ^4 theory

$$S_{st}(\varphi) \equiv \int d\mathbf{x}\,\mathcal{L} = \int d\mathbf{x}[(\nabla\varphi)^2 + m\varphi^2 - \frac{g}{4!}\varphi^4 + h\varphi]\,, \quad \varphi \equiv \varphi(\mathbf{x})$$

- Langevin equation

$$\partial_t\varphi(x) = \lambda\frac{\delta S_{st}(\varphi)}{\delta\phi(x)} + f(x), \;\; \langle f(x)f(x')\rangle = 2\lambda\delta(x-x'), \;\varphi(\mathbf{x}) \to \varphi(x), \;\; x \equiv \mathbf{x}, t$$

- Action of A and B models of critical dynamics

$$S(\varphi,\varphi') \equiv \int dx\,\mathcal{L} = \int dx\{\lambda\varphi'\varphi' + \varphi'[-\partial_t + \lambda\Delta - m]\varphi - \frac{g}{3!}\varphi'\varphi^3 + h\varphi\}$$

P.C. Hohenberg and B.I. Halperin, Rev. Mod. Phys. 49 (1977), 435

Systems with strong fluctuations

- Hydrodynamic turbulence

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}^{\mathbf{v}}\,, \quad \langle f^{\mathbf{v}}(x) f^{\mathbf{v}}(x')\rangle = \delta(t-t')D^{\mathbf{v}}(\mathbf{x},\mathbf{x}')$$

$$\partial_t \theta + (\mathbf{v}\cdot\nabla)\theta - u\nu\Delta\theta + H(\theta,\mathbf{v}) = \mathbf{f}^{\theta}\,, \quad \langle f^{\theta}(x) f^{\theta}(x')\rangle = \delta(t-t')D^{\theta}(\mathbf{x},\mathbf{x}')$$

- Stochastic MHD: $\theta \to \mathbf{b}$, $H(\theta,\mathbf{v}) \to H(\mathbf{b},\mathbf{v}) = -(\mathbf{b}\cdot\nabla)\mathbf{v}$ with $(\mathbf{b}\cdot\nabla)\mathbf{b}$ in NSE (Lorentz force)
- advection of passive scalar admixture, stochastic toy models: $H(\theta,\mathbf{v}) = 0$

Field theoretical description

- General stochastic differential equation with additive noise

$$\partial_t \phi(x) = V(x,\phi) + f(x), \quad \langle f(x) f(x') \rangle = D(x,x')$$

$$V(\phi) \sim \frac{\delta S(\phi)}{\delta \phi(x)}$$

- MSR mechanism
- de Dominicis - Jansen action functionl

$$S(\phi,\phi') = \frac{1}{2}\phi' D \phi' + \phi' \left[-\partial_t \phi + V(\phi) \right]$$

Systems with strong fluctuations

- generalization the mechanism to multiplicative noise:

$$f \quad \rightarrow \quad F(\phi)f \quad \rightarrow \quad e.g. \quad \phi f$$

- instructive to infer the generating functional from the mathematically well-defined setup of the stochastic problem
- evolution equations for probability density functions of the relevant random quantities
- continuous stochastic processes - the Fokker-Planck equation
- jump processes ((e.g. individuals of some population, molecules in chemical reaction)) - the master equation
- fundamental Chapman-Kolmogorov equation of Markov processes - the most important class of stochastic processes from the point of view of fluctuation kinetics

M.Hnatič, J. Honkonen, T. Lučivjanský: Advanced field-theoretical methods in stochastic dynamics and theory of developed turbulence. review article, soon to be publish

Systems with strong fluctuations

- jump processes
- reaction equation for two species A and B with the rate constants k_+ and k_-

$$s_A A + s_B B \underset{k_-}{\overset{k_+}{\rightleftharpoons}} r_A A + r_B B \,,$$

- s_A, s_B, r_A, r_B - coefficients describing in which proportions the agents react
- anihilation $A + A \to \varnothing$
- coagulation $A + A \to A$
- birth and death as reactions

$$A \underset{\gamma}{\overset{\lambda}{\leftrightharpoons}} 2A \,, \qquad A \xrightarrow{\beta} \varnothing$$

Verhulst (logistic) model

- 1845: P.V. Verhulst – description of population dynamics in closed environs
- The simplest kinetic description of the dynamics of the average particle numbers (meen field approximation) - rate equation for $n(t)$ - number of individuals at a given time instant t

$$\frac{dn}{dt} = -\beta n + \lambda n - \gamma n^2,$$

- β - rate of mortality , λ - rate of natality

Systems with strong fluctuations

- Master equations

$$\frac{dP_f(t)}{dt} = \sum_i [w(i \to f)P_i(t) - w(f \to i)P_f(t)]$$

- P_i - probabilty to find the system in state i
- $w(i \to f)$ - transition probability

- Master equations for Verhulst model

$$\frac{\mathrm{d}P(t,N)}{\mathrm{d}t} = \lambda(N-1)P(t,N-1) - \left(\beta N + \gamma N^2\right)P(t,N)$$

$$\frac{\mathrm{d}P(t,n)}{\mathrm{d}t} = [\beta(n+1) + \gamma(n+1)^2]P(t,n+1) + \lambda(n-1)P(t,n-1) - \left(\beta n + \lambda n + \gamma n^2\right)P(t,n)\,, \quad 0 < n < N\,,$$

$$\frac{\mathrm{d}P(t,0)}{\mathrm{d}t} = (\beta + \gamma)P(t,1)$$

Quantum - field models

- Doi mechanism: states in Fock space, creation and annihilation operators, Liouville operator
- Action of Verhulst model

$$S(\phi,\phi^\dagger) = \phi^\dagger\left[-\partial_t + (\lambda-\beta-\gamma)\right]\phi - \gamma\varphi^\dagger\phi^2 + \lambda{\phi^\dagger}^2\phi - \gamma{\phi^\dagger}^2\phi^2$$

M.Hnatič, J. Honkonen, T. Lučivjanský: Field Theoretic Technique for Irreversible Reaction Processes. Physics of Particles and Nuclei 44 (2) (2013), s.316-348

Quantum-field models

- Statistical correlations

$$\langle \phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = \frac{\int D\phi D\phi' \phi(x_1)\phi(x_2)\dots\phi(x_n)e^{S(\phi,\phi')}}{\int D\phi D\phi' e^{S(\phi,\phi')}}$$

$$\mathcal{S}_n(r) = \langle |\phi(\mathbf{r}+\mathbf{x},t) - \phi(\mathbf{x},t)|^n\rangle = \frac{\int D\phi D\phi' |\phi(r+x) - \phi(x)|^n e^{S(\phi,\phi')}}{\int D\phi D\phi' e^{S(\phi,\phi')}}$$

$$x \equiv \mathbf{x}, t; \qquad r = |\mathbf{r}|$$

Hydrodynamic turbulence: Stochastic and field theoretic description

- Stochastic NS equation

$$\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu_0 \Delta \mathbf{v} + \nabla p = \mathbf{f}^{\mathbf{v}} \,,\ \langle f^{\mathbf{v}}(x) f^{\mathbf{v}}(x') \rangle = \delta(t-t') D(\mathbf{x},\mathbf{x}') \,, \nabla \cdot \mathbf{v} = 0$$

- Field-theoretic model

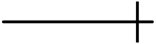
$$S(\mathbf{v},\mathbf{v}') = \frac{\mathbf{1}}{\mathbf{2}} \mathbf{v}' \cdot \mathbf{D} \cdot \mathbf{v} + \mathbf{v}' \cdot [-\partial_{\mathbf{t}} \mathbf{v} + \nu_{\mathbf{0}} \triangle \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v}]$$

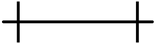
$$S_R(\mathbf{v},\mathbf{v}') = \frac{\mathbf{1}}{\mathbf{2}} \mathbf{v}' \cdot \mathbf{D} \cdot \mathbf{v} + \mathbf{v}' \cdot [-\partial_{\mathbf{t}} \mathbf{v} + \nu \mathbf{Z}_\nu \triangle \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v}]$$

$$\nu_0 = \nu Z_\nu, \;\; g_0 = g\mu^{2\epsilon} Z_g, \;\; Z_g = Z_\nu^{-3}, \; D_0 = g_0 \nu_0^3 = g\nu^3, \; d > 2$$

$$Z_\nu = 1 - \frac{cg}{2\epsilon} + O(g^2), \;\; c = \frac{(d-1)S_d}{4(2\pi)^d (d+2)} \,, S_d = 2\pi^{d/2}/\Gamma(d/2)$$

v_i $v_j \;=\; \langle v_i v_j \rangle_0 \equiv \Delta_{ij}^{vv}(\omega_k, \boldsymbol{k})$

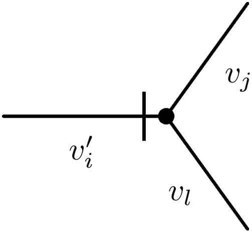
v_i  $v'_j \;=\; \langle v_i v'_j \rangle_0 \equiv \Delta_{ij}^{vv'}(\omega_k, \boldsymbol{k})$

v'_i  $v'_j \;=\; \langle v'_i v'_j \rangle_0 \equiv \Delta^{v'v'}(\omega_k, \boldsymbol{k})$

$$\Delta_{ij}^{\mathrm{vv}}(\mathbf{k}, \omega_k) = \frac{P_{ij}(\mathbf{k}) D(k)}{(i\omega_k + \nu k^2)(-i\omega_k + \nu k^2)}, \Delta_{ij}^{\mathrm{v'v'}}(\mathbf{k}, \omega_k) = 0,$$

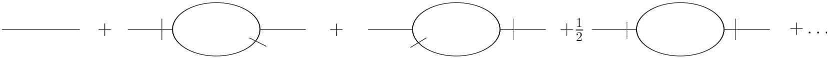
$$\Delta_{ij}^{\mathrm{vv'}}(\mathbf{k}, \omega_k) = \frac{P_{ij}(\mathbf{k})}{-i\omega_k + \nu k^2}, \Delta_{ij}^{\mathrm{v'v}}(\mathbf{k}, \omega_k) = \frac{P_{ij}(\mathbf{k})}{i\omega_k + \nu k^2}$$

Turbulence: Feynman graphs



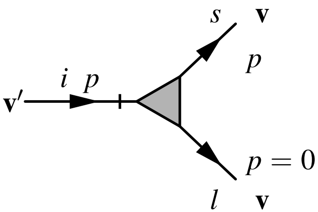
$$\equiv V_{ijl} = i(k_j\delta_{il} + k_l\delta_{ij})$$

- Vertex responsible for nonlinear interactions among velocity fluctuations
- Pair correlation function of velocity field with one-loop precision

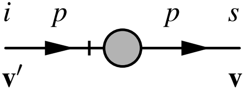


- Ward identity

$$\Gamma_{is\ l}(p,p,0) = k_l \frac{\partial}{\partial \omega} \Gamma_{is}(p)$$



$$= k_l \frac{\partial}{\partial \omega}$$



Renormalization group technique

- Renormalization group approach

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\gamma_\nu(g)\nu\frac{\partial}{\partial\nu}\right]\mathcal{S}_p(r)=0$$

$$\beta(g)=g(-2\epsilon+3\gamma_\nu)$$

$$s\frac{d\bar{g}}{ds}=\beta(\bar{g}),\ \ \bar{g}|_{s=1}=g,\qquad s\frac{d\bar{\nu}}{ds}=\gamma_\nu(\bar{g}),\ \ \bar{\nu}|_{s=1}=\nu$$

Renormalization group technique

- $s \equiv p/\mu$

$$\bar{\nu}(p) = \nu e^{\int_{\bar{g}}^{g} \frac{\gamma_\nu(x)}{\beta(x)} dx} = \left(\frac{g_0 \nu_0^3}{\bar{g} p^{2\epsilon}} \right)^{1/3}$$

- $\bar{g} \to g_*$ – perturbative calculations
- $\bar{\nu} \to \nu_*$
- $\gamma_\nu(g_*) = \frac{2\epsilon}{3}$

Renormalization group technique

- in the vicinity of the fixed point $r/l >> 1$ scaling:

$$\left[r\frac{\partial}{\partial r} + L\frac{\partial}{\partial L} + \Delta_p \right] \mathcal{S}_p(r) = 0$$

- solution:

$$\mathcal{S}_p(r) = (\mathcal{E}r)^{\Delta_p} f_p(r/L)$$

- $\Delta_p = p[\gamma_\nu - 1] = p[2\epsilon/3 - 1] \Rightarrow \epsilon = 2, \Delta_p = p[\gamma_\nu - 1] = p/3$

$$\mathcal{S}_2(r) = C_k(\mathcal{E}r)^{2/3}$$

Kolmogorov constant and skewness factor

- Perturbative calculation of quantity A - anomalous exponents, amplitudes, fixed points
- ϵ - expansion

$$A(\epsilon, d) = \sum_{k=0}^{\infty} A_k(d)\epsilon^k$$

- $A_k(d)$ - singularities at dimension $d = 2$ – Laurent series

$$A_k(d) = \sum_{l=0}^{\infty} a_{kl}\,\Delta^{l-k}, \qquad (d-2) \equiv 2\Delta$$

- double expansion in ϵ, Δ at fixed ζ

$$A(\epsilon, d) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \zeta^k a_{kl}\,\Delta^l\,, \qquad \zeta \equiv \epsilon/\Delta$$

Kolmogorov constant and skewness factor

- Definition of Kolmogorov constant

$$\mathcal{S}_2(r) = \left\langle |\mathbf{v}_r(\mathbf{r}+\mathbf{x}) - \mathbf{v}_r(\mathbf{x})|^2 \right\rangle = C_k \mathcal{E}^{2/3} r^{2/3}$$

$$\mathcal{S}_3(r) = \left\langle |\mathbf{v}_r(\mathbf{r}+\mathbf{x}) - \mathbf{v}_r(\mathbf{x})|^3 \right\rangle = -\frac{12}{d(d+2)} \mathcal{E} r$$

$$\mathbf{v}_r(\mathbf{x})| = (\mathbf{v}(\mathbf{x}) \cdot \mathbf{r})/r, \qquad r \equiv |\mathbf{r}|$$

- Definition of skewness factor

$$\mathcal{SF} = \mathcal{S}_3(r)/\mathcal{S}_2^{3/2}(r)$$

$$C_k = \left[-\frac{12}{d(d+2)\mathcal{SF}} \right]^{2/3}$$

Kolmogorov constant and skewness factor

- Result

$$C_k \approx 1.889, (1.47) \qquad exp. \approx 2.01$$

$$\mathcal{SF} \approx -0.308, (-0.45) \qquad exp. \approx -0.28$$

L.Ts. Adzhemyan, M.Hnatich, and J.Honkonen, Eur. Phys. J. B **73** (2010), 275

Operator product expansion

- Structure functions

$$\mathcal{S}_p(r) = (\mathcal{E}r)^{p/3} f_p(r/L)$$

- Wilson operator product expansion:

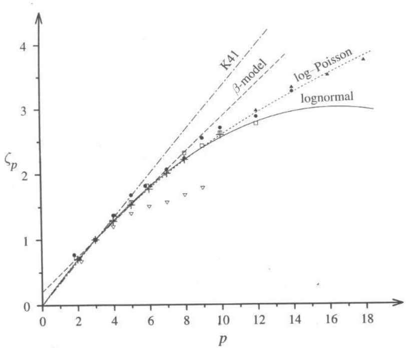
$$f_p(r/L) = \sum_{i=1}^{\infty} C_i^p (r/L)^{\Delta_i^p}, \quad r/L << 1$$

- $\Delta_i^p > 0 \Rightarrow f_p$ is a regular function at $L \to \infty \Rightarrow$ corrections to the leading anomalies calculated in the framework of canonical RG approach
- in the theory of turbulence the situation is quite different: critical exponents of composite operators Δ_i^p which describe velocity gradients are negative due to strong fluctuations of these gradients (intermittent behaviour)
- intermittency means that statistical properties of the turbulent velocity field are dominated by rare spatiotemporal configurations, in which the regions with strong turbulent activity have exotic fractal geometry and are embedded into the vast regions with regular laminar flow

Anomalous multiscaling

- Dangerous composite operators in the SNS model occur only for finite values of the RG expansion parameter ϵ
- Dangerous operators enter into the operator product expansions in the form of infinite families with the spectrum of critical dimensions unbounded from below, and the analysis of the large L behaviour implies the summation of their contributions
- This is clearly not a simple problem and it requires considerable improvement of the present technique
- consequence: anomalous multiscaling $\Leftrightarrow$ intermittency $\Leftrightarrow$ multifractality
- this is an open problem in theory of developed turbulence

Intermittency



Obr. : Intermittency

$$S_p(r) = \mathcal{E}^{p/3} r^{p/3} f_p(r/L), \qquad f_p(r/L) \sim (r/L)^{-\gamma_p}, \qquad \zeta_p = p/3 - \gamma_p$$

Passive scalar

- Advection of passive scalar field by turbulent flow
- Stochastic equation

$$\partial_t\theta + (\mathbf{v}\cdot\nabla)\theta = \nu_0\triangle\theta + f$$

$$\langle f(x)f(x')\rangle = \delta(t-t')\,C(\mathbf{x}-\mathbf{x}'/L)$$

$$\langle \mathbf{v}(x)\mathbf{v}(x')\rangle = D_{\mathbf{v}}(\mathbf{x}-\mathbf{x}')$$

- action functional

$$S(\theta,\theta',\mathbf{v}) = \theta' D_\theta \theta'/2 + \theta'[-\partial_t + \nu_0\triangle - (\mathbf{v}\cdot\nabla)]\theta - \mathbf{v}\cdot D_{\mathbf{v}}^{-1}\cdot\mathbf{v}/2$$

Structure functions

- Solution of basic RG equation for SF $\mathcal{S}_{2p}$
- Their asymptotic behavior for $r/l >> 1$ and any fixed r/L is given by IR stable fixed points of the RG equations :

$$\mathcal{S}_{2p}(r) \sim r^{p(2-\epsilon)} f_{2p}(r/L), \quad r/l >> 1$$

Structure functions: OPE

- Operator product expansion

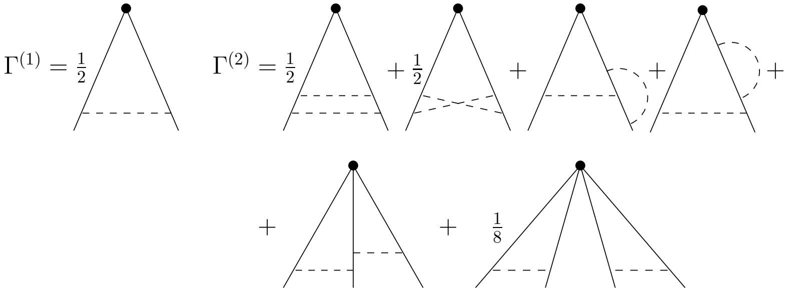
$$f_{2p}(r/L) = \sum_{F_k} C_{F_k}(r/L)\,(r/L)^{\Delta_k}, \;\; r/L \to 0$$

- The leading composite operators

$$F_s = (\partial_i \theta \partial_i \theta)^s \qquad \Delta_s = -s\gamma_\nu^* + \gamma_{F_s}^*, \quad s = 1 \ldots p$$

- $\Delta_s < 0 \Rightarrow$ anomalous scaling
- Multiplicative renormalization of F: $F_s = Z_{sl} F_l^R$

- Matrix of renormalization constants Z_{sl} are determined up to two loop approximation by the divergent parts of Feynman diagrams:



Critical dimensions

- Anomalous $\gamma^*_{F_s}$ and critical dimensions Δ_s are determined via eigenvalues of Z_{sl}
- Complete two-loop calculation of the critical dimensions of the composite operators F_s for arbitrary values of s, d:

$$\Delta_s = \Delta_s^{(1)}\epsilon + \Delta_s^{(2)}\epsilon^2$$

$$\Delta_s^{(1)} = \frac{-s(s-1)}{(d+2)} \qquad \Delta_s^{(2)} < 0$$

  - all Δ_s are negative already at small ϵ
  - Infinity series of OPE for SF S_{2p} truncate at p and CO F_p gives leading singular contribution Δ_p into asymptotic behaviour of scaling function at $r/L \ll 1$

Conclusions

- QFT - effective approach for investigation of the problems of stochastic dynamics and developed turbulence
- feedback for development of QFT methods - e.g. double (triple) expansion
- improvement of multi-loop computational schemes

Thank you for your attention

SPASIBO

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